

1 Supplementary material

Proof of Lemma 6. It is straightforward to prove that $\alpha + \omega_i$ is feasible for i , or $\gamma_0^i \leq \alpha + \omega_i \leq \delta^i$:

$$\begin{aligned}\alpha &\leq \delta^{i+1} \implies \alpha + \omega_i \leq \delta^{i+1} + \omega_i = \delta^i. \\ \alpha &\geq \gamma_0^{i+1} \geq q - \omega_{P_{i+1}} \implies \alpha + \omega_i \geq q - \omega_{P_{i+1}} + \omega_i = q - \omega_{P_i}.\end{aligned}$$

Since $\alpha + \omega_i \geq 0$, it follows that $\alpha + \omega_i \geq \max(q - \omega_{P_i}, 0) = \gamma_0^i$.

a) If $\alpha < \gamma_0^i$, every $T \subset P_{i+1}$ with $\omega_T \geq q - \alpha$ satisfies $i \in T$. Then:

$$\begin{aligned}b(i+1, \alpha) &= \max_{T \subset P_{i+1}; \omega_T \geq q - \alpha} (1 - d_T) = \max_{T \subset P_{i+1}; i \in T, \omega_T \geq q - \alpha} (1 - d_T) \\ &= \max_{T \subset P_i; \omega_T \geq q - \alpha - \omega_i} (1 - d_T) - d_i = b(i, \alpha + \omega_i) - d_i.\end{aligned}$$

b) If $\alpha \geq \gamma_0^i$, $b(i, \alpha)$ is well defined and

$$\begin{aligned}b(i+1, \alpha) &= \max_{T \subset P_{i+1}; \omega_T \geq q - \alpha} (1 - d_T) \\ &= \max \left\{ \max_{T \subset P_{i+1}; i \notin T, \omega_T \geq q - \alpha} (1 - d_T), \max_{T \subset P_{i+1}; i \in T, \omega_T \geq q - \alpha} (1 - d_T) \right\} \\ &= \max \left\{ \max_{T \subset P_i; \omega_T \geq q - \alpha} (1 - d_T), \max_{T \subset P_i; \omega_T \geq q - \alpha - \omega_i} (1 - d_T) - d_i \right\} \\ &= \max \{ b(i, \alpha), b(i, \alpha + \omega_i) - d_i \}.\end{aligned}$$

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Proof of Lemma 7. Let $\sigma^i \in \Sigma^i$ such that $\sigma^i > \omega_i$. The reader can check that $\gamma^{i+1} \leq \sigma^i - \omega_i \leq \delta^{i+1}$, hence $\sigma^i - \omega_i$ is a feasible value of α for player $i+1$. Note that for $b(i, \sigma^i) > 0$, homogeneity implies $\sigma^i - \omega_i \geq \omega_{i+1}$.

We now prove (2). Under Lemma 6a), (2) is true when $\sigma^i - \omega_i < \gamma_0^i$. Assume then $\sigma^i - \omega_i \geq \gamma_0^i$.

Since $\gamma^i \leq \sigma^i - \omega_i \leq \delta^{i+1}$, it is the case that for any $\tau^i \in T^i$

$$\frac{b(i, \tau^i)}{\tau^i} \geq \frac{b(i, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

Re-arranging terms,

$$\begin{aligned}
b(i, \sigma^i - \omega_i) &\leq \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) \\
&= b(i, \sigma^i) - \left[b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i) \right] \\
&\leq b(i, \sigma^i) - d_i.
\end{aligned}$$

Hence, (2) follows under lemma 6b). To show (3), we replace $\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}$ by $\frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i}$ and use $d_i \leq d_i^*$. We distinguish two subcases:

1. If $b(i, \tau^i) \geq 0$, then $d_i^* = b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)$.

$$\begin{aligned}
\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} &= \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \\
&\geq \frac{b(i, \sigma^i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^i - \omega_i} \\
&= \frac{b(i, \tau^i)}{\tau^i}
\end{aligned}$$

with strict inequality iff $d_i < d_i^*$.

2. If $b(i, \tau^i) < 0$, then $d_i^* = b(i, \sigma^i)$ and thus

$$\frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i} = \frac{b(i, \sigma^i) - d_i}{\sigma^i - \omega_i} \geq 0$$

with strict inequality iff $d_i < d_i^*$.

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Proof of Lemma 8. Let α such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 6, either $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$ or $b(i+1, \alpha) = b(i, \alpha)$. We have to prove that

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

If $b(i+1, \alpha) = b(i, \alpha)$, the result follows from (3).

If $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$, we have three cases:

1. If $\sigma^i \leq \delta^{i+1}$, then

$$\begin{aligned} \frac{b(i+1, \alpha)}{\alpha} &= \frac{b(i, \alpha + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\alpha} \leq \frac{\frac{b(i, \sigma^i)}{\sigma^i}(\alpha + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\alpha} \\ &= \frac{b(i, \sigma^i)}{\sigma^i} \stackrel{(\text{Lemma 7})}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}. \end{aligned}$$

2. If $\sigma^i > \delta^{i+1}$ and $b(i, \tau^i) \geq 0$ for some/all $\tau^i \in T^i$, then either $\frac{b(i, \alpha + \omega_i)}{\alpha + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i}$ (if $\alpha + \omega_i \leq \delta^{i+1}$) or $b(i, \alpha + \omega_i) = b(i, \sigma^i)$ (if $\alpha + \omega_i > \delta^{i+1}$, by Corollary 3).

If $\alpha + \omega_i \leq \delta^{i+1}$,

$$\begin{aligned} \frac{b(i+1, \alpha)}{\alpha} &= \frac{b(i, \alpha + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\ &\leq \frac{\frac{\alpha + \omega_i}{\tau^i} b(i, \tau^i) - \frac{\sigma^i}{\tau^i} b(i, \tau^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} \\ &= \frac{b(i, \tau^i)}{\tau^i} \stackrel{(\text{Lemma 7})}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}. \end{aligned}$$

If $\alpha + \omega_i > \delta^{i+1}$,

$$\frac{b(i+1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\alpha} = \frac{\sigma^i - \omega_i}{\alpha} \frac{b(i, \tau^i)}{\tau^i}.$$

If $b(i, \sigma^i) > 0$, corollary 3 implies $\sigma^i = \delta^{i+1} + 1$. Then $\alpha + \omega_i > \delta^{i+1}$ implies $\alpha + \omega_i \geq \delta^{i+1} + 1 = \sigma^i$, or $(\sigma^i - \omega_i) / \alpha \leq 1$. If $b(i, \sigma^i) = 0$, $b(i, \tau^i) = 0$, implying $\frac{\sigma^i - \omega_i}{\alpha} \frac{b(i, \tau^i)}{\tau^i} = \frac{b(i, \tau^i)}{\tau^i}$. In either case,

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i, \tau^i)}{\tau^i} \stackrel{(\text{Lemma 7})}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

3. If $\sigma^i > \delta^{i+1}$ and $b(i, \tau^i) < 0$ for some/all $\tau^i \in T^i$, then either $b(i, \alpha + \omega_i) < 0$ (if $\alpha + \omega_i \leq \delta^{i+1}$) or $b(i, \alpha + \omega_i) = b(i, \sigma^i)$ (if $\alpha + \omega_i > \delta^{i+1}$, by Corollary 3).

If $b(i, \alpha + \omega_i) < 0$,

$$\frac{b(i+1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} < -\frac{d_i}{\alpha} \leq 0 \stackrel{\text{(Lemma 7)}}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

If $b(i, \alpha + \omega_i) = b(i, \sigma^i)$,

$$\frac{b(i+1, \alpha)}{\alpha} = \frac{b(i, \alpha + \omega_i) - d_i}{\alpha} = 0 \stackrel{\text{(Lemma 7)}}{\leq} \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}.$$

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Proof of Lemma 9. a) Let $S \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$. We have to prove $b(i+1, \sigma^i - \omega_i) = 1 - d_{S \cup \{i\}}$. Using (2),

$$b(i+1, \sigma^i - \omega_i) = b(i, \sigma^i) - d_i = 1 - d_S - d_i = 1 - d_{S \cup \{i\}}.$$

b) Since $i \in S$, $b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i$, or

$$b(i, \sigma^{i+1} + \omega_i) = b(i+1, \sigma^{i+1}) + d_i. \quad (4)$$

Let $\sigma^i > \omega_i$. We have shown that $\sigma^i - \omega_i \in \Sigma^{i+1}$, thus

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i+1, \sigma^i - \omega_i)}{\sigma^i - \omega_i}. \quad (5)$$

1. If $\sigma^i \leq \delta^{i+1}$ for some $\sigma^i \in \Sigma^i$, it follows from (5) and (3) that

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \sigma^i)}{\sigma^i}. \quad (6)$$

Then

$$\begin{aligned} \frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} &\stackrel{(4)}{=} \frac{b(i+1, \sigma^{i+1}) + d_i}{\sigma^{i+1} + \omega_i} = \frac{b(i+1, \sigma^{i+1}) + \frac{\omega_i}{\sigma^i} b(i, \sigma^i)}{\sigma^{i+1} + \omega_i} \\ &\stackrel{(6)}{=} \frac{\frac{\sigma^{i+1}}{\sigma^i} b(i, \sigma^i) + \frac{\omega_i}{\sigma^i} b(i, \sigma^i)}{\sigma^{i+1} + \omega_i} = \frac{b(i, \sigma^i)}{\sigma^i}. \end{aligned}$$

Hence $\sigma^{i+1} + \omega_i \in \Sigma^i$ and $b(i, \sigma^{i+1} + \omega_i) = b(i+1, \sigma^{i+1}) + d_i = 1 - d_{S \cap P_i}$.

2. If $\sigma^i > \delta^{i+1}$ for all $\sigma^i \in \Sigma^i$, $\delta^{i+1} + 1$ always belongs to Σ^i .

Suppose $S \cap P_i \notin \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$ for all $\sigma^i \in \Sigma^i$. Then it must be the case that for any σ^i either $\omega_{S \cap P_i} < q - \sigma^i$, or $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal.

Suppose $\omega_{S \cap P_i} < q - \sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $\delta^{i+1} + 1 \in \Sigma^i$, it follows from Lemma 2 that $\omega_{S \cap P_i} < q - \delta^i$. But then $\omega_{S \cap P_i} + \omega_{N \setminus P_i} = \omega_{S \cap P_{i+1}} + \omega_{N \setminus P_{i+1}} < q$, contradicting the assumption that $\omega_{S \cap P_{i+1}} \geq q - \sigma^{i+1}$.

Suppose $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i} < 1 - d_T$ for some $\sigma^i \in \Sigma^i$ and $T \subset P_i$ with $\omega_T \geq q - \sigma^i$.

If $\sigma^{i+1} + \omega_i > \delta^{i+1}$,

$$b(i, \sigma^{i+1} + \omega_i) \stackrel{\text{(Lemma 2)}}{=} b(i, \sigma^i) > 1 - d_{S \cap P_i} = b(i+1, \sigma^{i+1}) + d_i$$

contradicting (4).

If $\sigma^{i+1} + \omega_i \leq \delta^{i+1}$, $\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i}$. There are two possibilities:

- If $b(i, \tau^i) \geq 0$, it follows from (5) and (3) that

$$\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i, \tau^i)}{\tau^i}. \quad (7)$$

Then

$$\begin{aligned} \frac{b(i, \tau^i)}{\tau^i} (\sigma^{i+1} + \omega_i) &\geq b(i, \sigma^{i+1} + \omega_i) \stackrel{(4)}{=} b(i+1, \sigma^{i+1}) + d_i = \\ &= \frac{b(i, \tau^i)}{\tau^i} \sigma^{i+1} + b(i, \sigma^i) - (\sigma^i - \omega_i) \frac{b(i, \tau^i)}{\tau^i} \end{aligned}$$

implying $\frac{b(i, \tau^i)}{\tau^i} \geq \frac{b(i, \sigma^i)}{\sigma^i}$, which is not possible because $\sigma^i > \delta^{i+1}$ for all $\sigma^i \in \Sigma^i$ implies $\frac{b(i, \tau^i)}{\tau^i} < \frac{b(i, \sigma^i)}{\sigma^i}$. Hence, this subcase is void.

- If $b(i, \tau^i) < 0$, it follows from (5) and (3) that $\frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} = 0$.

Then

$$b(i, \sigma^{i+1} + \omega_i) = b(i+1, \sigma^{i+1}) + d_i = b(i, \sigma^i).$$

Hence $b(i, \sigma^i) = 1 - d_{S \cap P_i}$ and the result follows.

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Proof of Lemma 10. a) Let $\sigma^{i+1} \in \Sigma^{i+1}$ and $\tau^i \in T^i$. We need to prove that $b(i, \sigma^{i+1})$ exists and $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$. This will be due to player $i+1$ having the option of setting $\alpha = \sigma^i$ (if $\sigma^i \leq \delta^{i+1}$) or $\alpha = \tau^i$ (if $\sigma^i > \delta^{i+1}$) without buying player i 's votes. We examine each case in turn:

1. If $\sigma^i \leq \delta^{i+1}$, then $d_i > \frac{\omega_i b(i, \sigma^i)}{\sigma^i}$.

Since $\sigma^i \leq \delta^{i+1}$, $b(i+1, \sigma^i)$ exists. Moreover, lemma 6b) implies

$$b(i+1, \sigma^i) \geq b(i, \sigma^i). \quad (8)$$

In principle, there are three possibilities for σ^{i+1} : either $\sigma^{i+1} < \gamma_0^i$, or $\sigma^{i+1} \geq \gamma_0^i$ and $b(i, \sigma^{i+1}) \leq b(i, \sigma^{i+1} + \omega_i) - d_i$, or $\sigma^{i+1} \geq \gamma_0^i$ and $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$. We will show that the first two possibilities lead to a contradiction. In both cases, Lemma 6 implies

$$b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i. \quad (9)$$

From (9) we can deduce:

$$\begin{aligned} \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\ &< \frac{b(i, \sigma^{i+1} + \omega_i) - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\sigma^{i+1}} \\ &\leq \frac{\frac{(\sigma^{i+1} + \omega_i) b(i, \sigma^i)}{\sigma^i} - \frac{\omega_i b(i, \sigma^i)}{\sigma^i}}{\sigma^{i+1}} \\ &= \frac{b(i, \sigma^i)}{\sigma^i} \stackrel{(8)}{\leq} \frac{b(i+1, \sigma^i)}{\sigma^i}. \end{aligned}$$

which contradicts that $\sigma^{i+1} \in \Sigma^{i+1}$. Thus, $\sigma^{i+1} \geq \gamma_0^i$ (i.e. $b(i, \sigma^{i+1})$ does exist) and $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$. We conclude then that $i \notin S$ for all $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$.

2. If $\sigma^i > \delta^{i+1}$, then $d_i > b(i, \sigma^i) - \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)$.

Under Lemma 6b):

$$b(i+1, \tau^i) = \max \{b(i, \tau^i), b(i, \tau^i + \omega_i) - d_i\} \geq b(i, \tau^i). \quad (10)$$

Suppose $b(i, \sigma^{i+1})$ does not exist (i.e. $\sigma^{i+1} < \gamma_0^i$), or $b(i, \sigma^{i+1})$ exists and $b(i, \sigma^{i+1}) \leq b(i, \sigma^{i+1} + \omega_i) - d_i$. In both cases, under Lemma 6,

$$b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i. \quad (11)$$

We will prove that (11) leads to a contradiction, so that $b(i, \sigma^{i+1})$ exists and $b(i, \sigma^{i+1}) > b(i, \sigma^{i+1} + \omega_i) - d_i$, which implies $i \notin S$ for all $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ as desired.

We have two cases:

- If $\sigma^{i+1} + \omega_i \leq \delta^{i+1}$. Then $\frac{b(i, \sigma^{i+1} + \omega_i)}{\sigma^{i+1} + \omega_i} \leq \frac{b(i, \tau^i)}{\tau^i}$ and

$$\begin{aligned} \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\ &< \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} \\ &\leq \frac{\frac{\sigma^{i+1} + \omega_i}{\tau^i} b(i, \tau^i) - \frac{\sigma^i}{\tau^i} b(i, \tau^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} \\ &= \frac{b(i, \tau^i)}{\tau^i} \stackrel{(10)}{\leq} \frac{b(i+1, \tau^i)}{\tau^i} \end{aligned}$$

which is a contradiction.

- If $\sigma^{i+1} + \omega_i > \delta^{i+1}$, then under Corollary 3, $b(i, \sigma^{i+1} + \omega_i) = b(i, \sigma^i)$. If $b(i, \sigma^i) > 0$, $\sigma^i = \delta^{i+1} + 1$ and $\sigma^{i+1} + \omega_i \geq \sigma^i$, which implies $(\sigma^i - \omega_i) / \sigma^{i+1} \leq 1$. If $b(i, \sigma^i) = 0$, $b(i, \tau^i) = 0$. Hence:

$$\begin{aligned} \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\ &< \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \sigma^i) + \frac{\sigma^i - \omega_i}{\tau^i} b(i, \tau^i)}{\sigma^{i+1}} \\ &= \frac{\sigma^i - \omega_i}{\sigma^{i+1}} \frac{b(i, \tau^i)}{\tau^i} \leq \frac{b(i, \tau^i)}{\tau^i} \stackrel{(10)}{\leq} \frac{b(i+1, \tau^i)}{\tau^i} \end{aligned}$$

which is a contradiction.

b) Recall that we assumed $b(i, \sigma^i) \geq 0$ for all $\sigma^i \in \Sigma^i$. Thus, $b(i, \tau^i) < 0$ for some $\tau^i \in T^i$ implies $\sigma^i > \delta^{i+1}$. Under Corollary 3, this means $b(i, \sigma^i) = b(i, \delta^i)$. Let α be such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 6, we have two cases:

1. $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$. Then

$$b(i+1, \alpha) < b(i, \alpha + \omega_i) - b(i, \delta^i).$$

Since $\alpha + \omega_i \leq \delta^i$, $b(i, \alpha + \omega_i) \leq b(i, \delta^i)$ and thus $b(i+1, \alpha) < 0$.

2. $b(i+1, \alpha) = b(i, \alpha)$. Then $\gamma_0^i \leq \alpha \leq \delta^{i+1}$ and

$$\frac{b(i+1, \alpha)}{\alpha} \leq \frac{b(i, \tau^i)}{\tau^i} < 0$$

and thus $b(i+1, \alpha) < 0$.

Since $b(i+1, \alpha) < 0$ for all α , we conclude $b(i+1, \sigma^{i+1}) < 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and thus by Lemma 5 all the players get zero. ■

Proof of Proposition 2. We proceed by backwards induction on i . For $i = n$, its strategy is clearly optimal.

Assume now the result is true for $\mathbb{B}(d, i+1)$ and moreover assume the following two conditions hold:

Condition 1 The formed coalition maximizes the benefit per vote, that is

$$S \cap P_{i+1} \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$$

for some $\sigma^{i+1} \in \Sigma^{i+1}$. (This condition holds trivially for $i+1 = n$ because $\Sigma^n = \{\omega_n\}$).

Condition 2 The above S and σ^{i+1} are such that $S \cap P_{i+1}$ is one of the most favorable sets for player i (i.e. $i \notin S$ implies $i \notin T$ for all $T \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$). Among them, it is one of the most favorable to player $i-1$, etc. (This condition holds for $i+1 = n$ because $\Sigma^n = \{\omega_n\}$ and n applies the tie-breaking rule).

We check that this remains true for $\mathbb{B}(d, i)$. Let $\tau^i \in T^i$. We have two cases:

1. If $\sigma^i > \omega_i$ for all $\sigma^i \in \Sigma^i$, then it is straightforward to check that player i obtains strictly less than d_i^* by forming coalition. If i demands d_i^* , there is a coalition $S \subset P_i$ such that $S \cup \{i\} \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ for $\sigma^{i+1} = \sigma^i - \omega_i \in \Sigma^{i+1}$ (Lemma 9a). The induction hypothesis (Conditions 1 and 2) implies that d_i^* will be accepted. Assume player i deviates by demanding $d_i > d_i^*$. If $b(i, \tau^i) \geq 0$, under Lemma 10a) player i does not belong to any coalition in $\arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ for any $\sigma^{i+1} \in \Sigma^{i+1}$ and its final payoff is zero under the induction hypothesis (Condition 1). If $b(i, \tau^i) < 0$, under Lemma 10b), its final payoff is zero.

Moreover, Conditions 1 and 2 hold for i . Condition 1 follows from Lemma 9b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the player $j > i$ that eventually forms coalition.

2. If $\omega_i \in \Sigma^i$, then $1 - d_S = b(i, \omega_i) = d_i^*$ for all $S \in \arg \max_{T \subset P_i: \omega_T \geq q - \omega_i} (1 - d_T)$. This means that if player i forms a winning coalition it obtains a final payoff of $b(i, \omega_i)$. Suppose player i deviates and demands $d_i > b(i, \omega_i)$. It is enough to check that $i \notin S$ for all $S \in \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$. Under the induction hypothesis applied to Condition 1, this means that player i will not be included in any eventual winning coalition, and its final payoff will be zero, while the original strategy yields a nonnegative payoff.

For constant-sum homogeneous games it is always the case that $\omega_i \leq \delta^{i+1}$, thus $b(i+1, \omega_i)$ is well defined. Under Lemma 6b),

$$b(i+1, \omega_i) = \max \{b(i, \omega_i), b(i, 2\omega_i) - d_i\} \geq b(i, \omega_i) \quad (12)$$

Suppose that $i \in S$ for some $S \in \arg \min_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} d_T$ and some $\sigma^{i+1} \in \Sigma^{i+1}$. This means

$$b(i+1, \sigma^{i+1}) = b(i, \sigma^{i+1} + \omega_i) - d_i$$

and hence

$$\begin{aligned} \frac{b(i+1, \sigma^{i+1})}{\sigma^{i+1}} &= \frac{b(i, \sigma^{i+1} + \omega_i) - d_i}{\sigma^{i+1}} \\ &< \frac{b(i, \sigma^{i+1} + \omega_i) - b(i, \omega_i)}{\sigma^{i+1}} \\ &\leq \frac{\frac{\sigma^{i+1} + \omega_i}{\omega_i} b(i, \omega_i) - b(i, \omega_i)}{\sigma^{i+1}} \\ &= \frac{(\sigma^{i+1} + \omega_i) b(i, \omega_i) - \omega_i b(i, \omega_i)}{\omega_i \sigma^{i+1}} \\ &= \frac{b(i, \omega_i)}{\omega_i} \stackrel{(12)}{\leq} \frac{b(i+1, \omega_i)}{\omega_i} \end{aligned}$$

which is a contradiction. This contradiction proves that $i \notin S$ for all $S \in \arg \min_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} d_T$, as desired.

We now check that Conditions 1 and 2 hold for i . If player i forms coalition, Condition 1 holds with $\sigma^i = \omega_i$, and Condition 2 holds because of the tie-breaking rule. If player i demands d_i^* so that S^* is induced, it must be the case that $\{\omega_i\} \not\subseteq \Sigma^i$. Hence, there exists $\sigma^i \in \Sigma^i$ with $\sigma^i > \omega_i$. Then, Condition 1 follows from Lemma 9b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the player that eventually forms coalition.

■

Proof of proposition 3. We proceed by backwards induction on i . We prove the following three hypotheses:

1. If $b(i, \sigma^i) < 0$, all players get zero in every SPE of $\mathbb{B}(d, i)$.

2. If $b(i, \sigma^i) > 0$, player i receives $d_i^* > 0$ in every SPE of $\mathbb{B}(d, i)$ and the coalition that forms satisfies $S \cap P_i \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.
3. If $b(i, \sigma^i) = 0$,
 - a) player i gets $d_i^* = 0$ in every SPE of $\mathbb{B}(d, i)$;
 - b) there is a SPE of $\mathbb{B}(d, i)$ in which a winning coalition forms;
 - c) if a winning coalition S forms, then $S \cap P_i \in \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

The induction hypothesis holds for player n . Now suppose it holds for player $i + 1$. Does it hold for player i ?

1. If $b(i, \sigma^i) < 0$, all players get zero (Lemma 5).
2. If $b(i, \sigma^i) > 0$, player i cannot get more than d_i^* by forming coalition. If player i demands more than d_i^* and $b(i, \tau^i) \geq 0$, we know from Lemma 10a) that $i \notin \arg \max_{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}} (1 - d_T)$ for all $\sigma^{i+1} \in \Sigma^{i+1}$. The induction hypothesis implies that player i gets zero. If player i demands more than d_i^* and $b(i, \tau^i) < 0$, we know from Lemma 10b) that player i gets zero.

Now we show that player i can get at least d_i^* . This is immediate if $\omega_i \in \Sigma^i$. Suppose $\omega_i \notin \Sigma^i$. Since $b(i, \sigma^i) > 0$, we know $d_i^* > 0$. The value of d_{i+1}^* induced by d_i^* may be strictly positive or 0. Suppose player i demands $d_i < d_i^*$. Then the corresponding value of d_{i+1}^* is strictly positive. Under Lemma 11, player i belongs to all coalitions associated with some element of Σ^{i+1} , and the induction hypothesis for $d_{i+1}^* > 0$ implies that player i gets d_i . Thus, the perfectness of the equilibrium implies that d_i^* is accepted (otherwise, player i would not have a best response).

Moreover, Lemma 9b), the induction hypothesis and the fact that d_i^* is accepted imply that the coalition that forms satisfies $S \cap P_i \in$

$$\arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T) \text{ for some } \sigma^i \in \Sigma^i.$$

3. If $b(i, \sigma^i) = 0$, then $d_i^* = 0$ and, moreover, $\alpha \in \Sigma^i$ if and only if $b(i, \alpha) = 0$.

a) It is trivial that player i gets $d_i^* = 0$. If $d_i > d_i^*$, the induction hypothesis implies that no coalition to which player i belongs will form.

b) There is an equilibrium of the subgame in which a coalition associated with $\sigma^i \in \Sigma^i$ forms. This is clearly the case for $\omega_i \in \Sigma^i$. Otherwise, it is optimal for player i to demand $d_i^* = 0$. Then $b(i+1, \sigma^{i+1}) = 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and the induction hypothesis implies that there is a SPE of $\mathbb{B}(i, d)$ in which a winning coalition is formed.

c) Assume a winning coalition S is formed with $S \cap P_i \notin \arg \max_{T \subset P_i: \omega_T \geq q - \sigma^i} (1 - d_T)$ for all $\sigma^i \in \Sigma^i$. This means that, for a given $\sigma^i \in \Sigma^i$, either $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal, or $\omega_{S \cap P_i} < q - \sigma^i$.

Assume first there exists $\sigma^i \in \Sigma^i$ such that $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal. Since $b(i, \sigma^i) = 0$, this means $d_{S \cap P_i} > 1$ and it cannot be optimal at any subgame to form S .

Assume now $\omega_{S \cap P_i} < q - \sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $b(i, \sigma^i) = 0$ and $b(i, \alpha)$ is nondecreasing in α , $\delta^i \in \Sigma^i$; thus $\omega_{S \cap P_i} < q - \delta^i$. This means $\omega_{S \cap P_i} + \omega_{S \cap (N \setminus P_i)} < q$. Thus, S is not a winning coalition.

■