

A non-cooperative approach to the ordinal Shapley rule*

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Abstract

In bargaining problems, a rule satisfies ordinal invariance if it does not depend on order-preserving transformations of the agent's utilities. In this paper, a simple non-cooperative game for three players, based on bilateral offers, is presented. The ordinal Shapley rule arises in subgame perfect equilibrium as the players have more time to reach an agreement.

Keywords: ordinal bargaining, Shapley-Shubik rule

1 Introduction

In bargaining problems, a rule satisfies ordinal invariance if it does not depend on order-preserving transformations of the agents' utilities. For two

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agents, Shapley (1969) shows that no efficient rule, apart from the dictatorial one, satisfies ordinal invariance. However, this negative result does not hold any more for more than two agents. Shubik (1982) first documents an efficient, symmetric, and ordinal invariant rule for three agents. Even though there is no reference on the origin of this rule in Shubik (1982), Pérez-Castrillo and Wettstein (2002, "An Ordinal Shapley value for ordinal environments", WP, forthcoming in JET, p.2) attribute it to Shapley (1969). Furthermore, Roth (1979, p. 72-73) mentions a three-player ordinal bargaining rule proposed by Shapley and Shubik in a 1974 working paper. Kibris (2004b) suggests that they probably are the same bargaining rule. Kibris (2004b) refers to it as the Shapley-Shubik rule.

Kibris (2001) describes a class of three-agent problems which generates all bargaining problems. On this class, the Shapley rule coincides with the Egalitarian rule (Kalai, 1977) and the Kalai-Smorodinsky rule (1975), and moreover it is the only symmetric member of a class of ordinal monotone path rules. Moreover, Kibris (2002) characterizes the Shapley rule using a weaker version of *Independence of Irrelevant Alternatives* (Nash, 1950). On the other hand, Safra and Samet (2004a) extend the Shapley rule for more than three players using constructions similar to O'Neill, Samet, Wiener and Winter (2004). Safra and Samet (2004b) provide yet another family of ordinal solutions.

Following a different approach, Pérez-Castrillo and Wettstein (2002) use the underlying physical environment generating the utility possibilities frontier. This allows them to define an ordinal extension for the Shapley value for an arbitrary number of players. We will call this value the ordinal Shapley value.

Finally, a mixed approach is given by Calvo and Peters (2005) who studies situations where there exist ordinal and cardinal players.

The definitions of these values are normative. An alternative approach is to propose non-cooperative games whose equilibria yield reasonable outcomes. This is the basis of the so-called Nash program, first suggested by

Nash (1953), also related to the theory of implementation.

A non-cooperative game yielding the ordinal Shapley value in subgame perfect equilibria for three players is presented in Pérez-Castrillo and Wettstein (2005).

One may wonder whether it is possible to find a similar result for the Shapley rule. In this paper, a non-cooperative game is presented whose unique subgame perfect equilibrium payoff allocation approaches the Shapley rule as the players have more time to reach an agreement.

Informally, the idea of the non-cooperative game is as follows: First, one of the players proposes to a second one a payoff allocation. This payoff allocation constitutes a pre-agreement between them. The third player can then choose one of these players and make him a counter-proposal, which in case of being accepted would cancel the pre-agreement. However, if the counter-proposal is rejected, the other player makes a last offer and the pre-agreement remains as a status quo in case of rejection. Additionally, each player has a veto option that can make the process to be repeated in the next round. If no agreement is reached after a pre-specified number of rounds, the disagreement payoff allocation is implemented.

As the number of round increases, there exists a subgame perfect equilibrium whose payoff allocation approaches the Shapley rule. Under reasonable assumptions on the behavior of the agents when they are indifferent (tie-breaking rules), this equilibrium is unique.

2 Preliminaries

Let $N = \{1, 2, 3\}$ be the set of agents. Given $x, y \in \mathbb{R}^N$, $x \leq y$ means $x_i \leq y_i$ for all $i \in N$, $x \ll y$ means $x_i < y_i$ for all $i \in N$, and $x < y$ means $x \leq y$ and $x \neq y$. Let Π be the set of all permutations π of N .

A pair $(S, d) \in 2^{\mathbb{R}^N} \times \mathbb{R}^N$ is a *bargaining problem* if $S \cap \{x \in \mathbb{R}^N : d \leq x\}$ is compact and d belongs to the interior of S . A point $x \in S$ is *Pareto optimal* if $y \leq x$ for all $y \in S$. Let $P(S)$ denote the set of Pareto optimal points in

S . A point $x \in S$ is *weakly Pareto optimal* if there is no $y \in S$ such that $x < y$. A point $x \in S$ is *individually rational* if $d \leq x$. Let $WP(S)$ denote the set of weakly Pareto optimal points in S .

A bargaining problem (S, d) is *strictly comprehensive* if $WP(S) = P(S)$ and for each $x \in S$, $y \leq x$ implies $y \in S$. Let \mathcal{B} denote the set of all strictly comprehensive bargaining problems.

For each $(S, d) \in \mathcal{B}$, $x, y \in \mathbb{R}^N$ and $N = \{i, j, k\}$, player i 's *aspiration payoff restricted to x_j and y_k* is

$$a_i(S, x_j + y_k) = \max \{s_i : (s_i, x_j, y_k) \in S\}$$

and her *aspiration point restricted to x_j and y_k* is $a(S, x_j + y_k) = (a_i(S, x_j + y_k), x_j, y_k)$.

Let $(S, d) \in \mathcal{B}$. Define $p^0(S, d) = d$. For each $t > 0$, there exists a unique $p^t(S, d) \in R^N$ such that

$$\begin{aligned} p^{t,12}(S, d) &\equiv (p_1^t(S, d), p_2^t(S, d), p_3^{t-1}(S, d)) \in P(S) \\ p^{t,13}(S, d) &\equiv (p_1^t(S, d), p_2^{t-1}(S, d), p_3^t(S, d)) \in P(S), \text{ and} \\ p^{t,23}(S, d) &\equiv (p_1^{t-1}(S, d), p_2^t(S, d), p_3^t(S, d)) \in P(S). \end{aligned}$$

The sequence $\{p^t(S, d)\}_{t \geq 0}$ is uniquely defined and it is convergent. Also, for each $i, j \in N$,

$$\lim_{t \rightarrow \infty} p^t(S, d) = \lim_{t \rightarrow \infty} p^{t,ij}(S, d)$$

A bargaining rule $F : \mathcal{B} \rightarrow R^N$ assigns each bargaining problem $(S, d) \in \mathcal{B}$ to a feasible point $F(S, d) \in S$. For each $(S, d) \in \mathcal{B}$, the Shapley bargaining rule, $Sh : \mathcal{B} \rightarrow R^N$ selects the limit of the sequence $\{p^t(S, d)\}_{t \geq 0}$ as the solution:

$$Sh(S, d) \equiv \lim_{t \rightarrow \infty} p^t(S, d).$$

3 Results

3.1 The non-cooperative game

There are exactly T negotiation rounds. If no agreement is reached at round $T + 1$, the disagreement payoff allocation d is implemented. At each round, the players play the roles of *first proposer*, *first responder*, and *pivot*. The first proposer proposes a payoff allocation $x \in S$. A round passes by if the first responder vetoes this proposal. In this case, in the next round the vetoer plays the role of pivot and the pivot chooses to be either the first proposer or the first responder. Once a proposal $x \in S$, if any, is accepted, it would be considered as a pre-agreement between the first proposer and the first responder. The pivot makes then a counter-proposal $y \in S$ to one of the other two players. Let i be this player and let j be the other one. Player i should choose between the pre-agreement x and the counter-proposal y . Two cases are possible:

1. If i chooses y , player j can still veto this decision. If j does not veto, y is implemented and the game finishes. If j vetoes, a round passes by.
2. If i chooses x , player j makes a last proposal $z \in S$. Player i should choose between x and z . The pivot can still veto this decision. If the pivot does not veto, the chosen allocation (x or z) is implemented and the game finishes. If the pivot vetoes, a round passes by.

In case of veto, in the next round the vetoer plays the role of pivot and player i chooses to be either the first proposer or the first responder.

In order to fully understand the non-cooperative game, a formal description is presented as follows. We denote the game as $B^t(\pi)$, where t is the number of rounds left (hence, we begin with $B^T(\pi)$) and $\pi \in \Pi$ is the order that specifies the roles: π_1 is the first proposer, π_2 is the first responder, and π_3 is the pivot.

The non-cooperative game is defined inductively on t . In $B^0(\pi)$, each player $i \in N$ receives d_i .

Assume $B^s(\pi)$ is defined for all $s < t$. Assume wlog that $\pi = [123]$, we define $B^t([123])$ as follows:

Player 1 proposes $x \in S$. Player 2 can veto or not veto.

If player 2 vetoes, player 3 chooses to play either $B^{t-1}([132])$ or $B^{t-1}([312])$. If player 2 does not veto, player 3 chooses $i \in \{1, 2\}$ and proposes $y \in S$. Let $j \equiv \{1, 2\} \setminus \{i\}$ be the other player.

Player i can choose x or y .

If player i chooses y , player j can veto or not veto. If player j vetoes, player i chooses to play either $B^{t-1}([i3j])$ or $B^{t-1}([3ij])$. If player j does not veto, y is implemented and the game finishes.

If player i chooses x , player j proposes $z \in S$. Player i can choose x or z . Let $q \in \{x, z\}$ be his choice. Player 3 can veto or not veto. If player 3 vetoes, player i chooses to play either $B^{t-1}([ij3])$ or $B^{t-1}([j3i])$. If player 3 does not veto, q is implemented and the game finishes.

In the following, we assume wlog that in the first round the first proposer is player 1, the first responder is player 2, and the pivot is player 3.

Theorem 3.1 *There exists a SP equilibrium for the non-cooperative game whose equilibrium payoff allocation is $p^{T,13}(S, d)$.*

In general, there can be more than one SP equilibrium. However, the above SP equilibrium is unique under the following Assumptions:

Assumption 1 If a player is indifferent between vetoing and not vetoing, he strictly prefers not to veto.

Assumption 2 If the pivot (say k) is indifferent when choosing i , and x_k is strictly less than the maximum x_k in SP equilibrium, then he strictly prefers to choose the first responder.

Assumption 1 follows from the fact that a veto implies a delay. Hence, it seems natural that, in case of indifference, a player would prefer to reach an agreement as soon as possible.

Assumption 2 has two justifications, one of them is ex ante and the other one is ex post. Ex ante, since x comes from the first proposer, and it is "bad" for the pivot (in the sense that there exist SP equilibria in which his share is bigger), it seems natural to renegotiate it with the other player. Ex post, the equilibria in which an indifferent pivot chooses the first proposer (and x_k is "low") yields worse outcomes for the proposer than those in which the first proposer chooses the first proposer.

Theorem 3.2 *Under Assumptions 1 and 2, $p^{T,13}(S, d)$ is the unique SP equilibrium payoff allocation for the non-cooperative game.*

An immediate corollary is the following:

Corollary 3.1 *Under Assumptions 1 and 2, as T increases the only equilibrium payoff allocation in the non-cooperative game approaches the payoff allocation given by the Shapley rule.*

3.2 Proof of Theorem 3.1 and Theorem 3.2

Recall $N = \{1, 2, 3\}$, 1 is the first proposer, 2 is the first responder, and 3 is the pivot. The proof is by induction on T . For $T = 0$, the result is trivially true. Assume the result is true for less than T rounds. The subgame that arises in the second round of the game with T rounds is strategically equivalent to the game with $T - 1$ rounds. Hence, the continuation payoff in the second round is known by the agents. Let $b^{T-1}(\pi)$ be the continuation payoff in the second round when the order is given by $\pi \in \Pi$. Under the induction hypothesis, $b^{T-1}(\pi) = p^{T-1, \pi_1 \pi_3}$.

Assume we are in SP equilibrium. Let b^T be the equilibrium payoff allocation.

Claim 1.1: Assume $x_3 < p_3^{T,12}$. In the subgame that begins when j chooses $z \in S$, the final payoff allocation is $p^{T-1,j3}$ if T is odd, and $p^{T-1,i3}$ if T is even.

Since $x_3 < p_3^{T,12}$, player i can induce a payoff allocation of $p^{T-1,i3}$ or $p^{T-1,j3}$ by choosing $q = x$, knowing that 3 is bound to veto this choice and hence force $B^{T-1}[ij3]$ or $B^{T-1}[ji3]$. Notice that 3 would get $p_3^{T-1,3} = p_3^{T,12}$ by vetoing and x by not vetoing. On the other hand, player 3 can assure himself a payoff of $p_3^{T-1,3}$ by vetoing any q . Hence, the equilibrium payoff for player j is at most

$$\begin{aligned} a_j \left(S, \max \left\{ p_i^{T-1,i3}, p_i^{T-1,j3} \right\} + p_3^{T-1,3} \right) &= \begin{cases} a_j \left(S, p_i^{T-1,j3} + p_3^{T-1,3} \right) & \text{if } T \text{ odd} \\ a_j \left(S, p_i^{T-1,i3} + p_3^{T-1,3} \right) & \text{if } T \text{ even} \end{cases} \\ &= \begin{cases} p_j^{T-1,j3} & \text{if } T \text{ odd} \\ p_j^{T-1,i3} & \text{if } T \text{ even.} \end{cases} \end{aligned}$$

If T is odd, player j can assure himself any $p_j^{T-1,j3} - \varepsilon_j$ for all $\varepsilon_j > 0$ by proposing $z = p^{T-1,j3} + (\varepsilon_i, -\varepsilon_j, \varepsilon_3) \in S$ for appropriate values of $\varepsilon_i > 0$ and $\varepsilon_3 > 0$. Hence, in SP equilibrium, player j gets at least $p_j^{T-1,j3}$. Since each player can assure $p^{T-1,j3} \in S$, this is the only possible final payoff in SP equilibrium.

If T is even, player j can assure himself any $p_j^{T-1,i3} - \varepsilon_j$ for all $\varepsilon_j > 0$ by proposing $z = p^{T-1,i2} + (\varepsilon_i, -\varepsilon_j, \varepsilon_3) \in S$ for appropriate values of $\varepsilon_i > 0$ and $\varepsilon_3 > 0$. Hence, in SP equilibrium, player j gets at least $p_j^{T-1,i3}$. Since each player can assure $p^{T-1,i3} \in P(S)$, this is the only possible final payoff in SP equilibrium.

Claim 1.2: Assume $x_3 \geq p_3^{T,12}$. In the subgame that begins when j chooses $z \in S$, the final payoff allocation is $a \left(S, x_i + p_3^{T,12} \right)$.

Since $x_3 \geq p_3^{T,12}$, player i can assure himself a payoff of x_i by choosing $q = x$, knowing that 3 would not veto this choice. Notice that 3 would get $p_3^{T-1,3} = p_3^{T,12}$ by vetoing and x by not vetoing. On the other hand, player 3 can assure himself a payoff of $p_3^{T,12}$ by vetoing any q . Hence, the equilibrium

payoff for player j is at most

$$a_j \left(S, x_i + p_3^{T,12} \right).$$

Moreover, player j can assure himself any $a_j \left(S, x_i + p_3^{T,12} \right) - \varepsilon_j$ for all $\varepsilon_j > 0$ by proposing $z = a \left(S, x_i + p_3^{T,12} \right) + (\varepsilon_i, -\varepsilon_j, \varepsilon_3) \in S$ for appropriate values of $\varepsilon_i > 0$ and $\varepsilon_3 > 0$. Hence, in SP equilibrium, player j gets at least $a_j \left(S, x_i + p_3^{T,12} \right)$.

Since each player can assure $a \left(S, x_i + p_3^{T,12} \right) \in P(S)$, this is the only possible final payoff allocation in SP equilibrium.

Claim 2.1: Assume $x_3 < p_3^{T,12}$. In the subgame that begins when 3 chooses $y \in S$, the final payoff is $p^{T-1,j3}$, if T is odd, and $p^{T-1,i3}$, if T is even.

Assume T is odd (even). Since $x_3 < p_3^{T,12}$, under Claim 1.1, player i can induce $p^{T-1,j3}$ ($p^{T-1,i3}$) by choosing x . On the other hand, if $y_j \geq p_j^{T-1,j}$ player j would not veto y and hence player i can induce y by choosing y . Hence, if $y_j \geq p_j^{T-1,j}$ and $y_i > p_i^{T-1,j3}$ ($p^{T-1,i3}$), then in SP equilibrium player i should choose y and hence y will be the final payoff allocation. Hence, if 3 induces y , his optimal final payoff allocation is

$$\begin{aligned} a \left(S, p_j^{T-1,j} + p_i^{T-1,j3} \right) &= a \left(S, p_j^{T-1,j3} + p_i^{T-1,j3} \right) = p^{T-1,j3} \\ (a \left(S, p_j^{T-1,j} + p_i^{T-1,i3} \right) &= a \left(S, p_j^{T-1,ij} + p_i^{T-1,ij} \right) = p^{T-1,ij}). \end{aligned}$$

We see now which is player 3's maximal payoff by not inducing y .

If $y_j \geq p_j^{T-1,j}$ and $y_i < p_i^{T-1,j3}$ ($p_i^{T-1,i3}$), then in SP equilibrium player i should choose x and hence $p^{T-1,j3}$ ($p^{T-1,i3}$) will be the final payoff allocation.

In case $y_j < p_j^{T-1,j}$. Player j would induce $p^{T-1,j3}$ ($p^{T-1,i3}$) in case y is chosen. In case x is chosen, under Claim 1.1 the final payoff allocation is $p^{T-1,j3}$ ($p^{T-1,i3}$) too. Hence, in any case (when $y_j < p_j^{T-1,j}$) the final payoff allocation is $p^{T-1,j3}$ ($p^{T-1,i3}$). This proves that, when T is odd, the final payoff is $p^{T-1,j3}$. Player 3 can induce either $p^{T-1,ij}$ or $p^{T-1,i3}$. But, when T is even, we have $p_3^{T-1,ij} < p_3^{T-1,i3}$ and hence player 3 will induce $p^{T-1,i3}$.

Claim 2.2: Assume $x_3 \geq p_3^{T,12}$. In the subgame that begins when 3 chooses $y \in S$,

- if $p_3^{T,12} < a_3 \left(S, x_i + p_j^{T,i3} \right)$, the final allocation is $a \left(S, x_i + p_j^{T,i3} \right)$
- if $p_3^{T,12} > a_3 \left(S, x_i + p_j^{T,i3} \right)$, the final allocation is $a \left(S, x_i + p_3^{T,12} \right)$.

Since $x_3 \geq p_3^{T,12}$, under Claim 1.2 player i can induce $a \left(S, x_i + p_3^{T,12} \right)$ by choosing x . On the other hand, if $y_j \geq p_j^{T-1,j} = p_j^{T,i3}$ player j would not veto y and hence i can induce y by choosing y . Hence, if $y_j \geq p_j^{T,i3}$ and $y_i < x_i$, the final payoff allocation should be $a \left(S, x_i + p_3^{T,12} \right)$ because player i will choose x . This means that 3 can induce $a \left(S, x_i + p_3^{T,12} \right)$ by choosing an appropriate y . In this case, 3 receives $p_3^{T,12}$.

If $y_j \geq p_j^{T,i3}$ and $y_i > x_i$, the final payoff allocation should be y . In this case, 3 receives at most $a_3 \left(S, x_i + p_j^{T,i3} \right)$. Moreover, for all $\varepsilon_3 > 0$, player 3 can assure himself

$$a_3 \left(S, x_i + p_j^{T,i3} \right) - \varepsilon_3$$

by proposing $y = a \left(S, x_i + p_j^{T,i2} \right) + (0, \varepsilon_i, -\varepsilon_3) \in S$ for an appropriate value of $\varepsilon_i > 0$.

If $y_j < p_j^{T,i3}$ then player i can either induce $b^{T-1} [ij3]$ or $b^{T-1} [3ji]$ by choosing y . In either case, player 3 would receive not more¹ than $p_3^{T,13}$ after a veto and hence it is optimal for 3 to induce $a \left(S, x_i + p_3^{T,12} \right)$.

As a conclusion, if $p_3^{T,12} < a_3 \left(S, x_i + p_j^{T,i3} \right)$, then player 3 would induce $a \left(S, x_i + p_j^{T,i3} \right)$. If $p_3^{T,12} > a_3 \left(S, x_i + p_j^{T,i3} \right)$, then 3 would induce $a \left(S, x_i + p_3^{T,12} \right)$.

Claim 3.2: Assume $x_3 \geq p_3^{T,12}$ and $x \neq p^{T,12}$. In the subgame that begins when 3 chooses $i \in \{1, 2\}$, we have

1. $i = \arg \max_{k \in \{1,2\}} a_2 \left(S, x_k + p_{k^c}^{T,k2} \right)$,
2. the final allocation is $a \left(S, x_i + p_j^{T,i3} \right)$, and

¹If T is odd, player i induces $b^{T-1} [3ji] = p^{T-1,3j}$ and $p_3^{T-1,j3} = p_3^{T,ij} = p_3^{T,12}$. If T is even, player i induces $b^{T-1} [ij3] = p^{T-1,ij} = p^{T-1,12}$ and $p_3^{T-1,13} < p_3^{T,12}$.

$$3. a_3 \left(S, x_i + p_j^{T,i3} \right) > p_3^{T,12}.$$

Under Claim 2.2, player 3 can get at least $p_3^{T,12}$. To improve this, he chooses the optimal i such that he gets $\max_{i \in \{1,2\}} a_3 \left(S, x_i + p_j^{T,i3} \right)$. Define $f, f^1, f^2 : \left\{ x \in S : x_3 \geq p_3^{T,12} \right\} \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} f^1(x) &= a_2 \left(S, x_1 + p_2^{T,13} \right) \\ f^2(x) &= a_2 \left(S, x_2 + p_1^{T,23} \right) \\ f(x) &= \max \{ f^1(x), f^2(x) \}. \end{aligned}$$

By choosing an optimal i , player 3 gets $f(x)$. Since f^1 is strictly decreasing in x_1 and f^2 is strictly decreasing in x_2 , f should reach a minimum when $x \in P(S)$, $x_3 = p_3^{T,12}$ and $f^1(x) = f^2(x)$, i.e.

$$\begin{aligned} a_3 \left(S, x_1 + p_2^{T,13} \right) &= a_3 \left(S, x_2 + p_1^{T,23} \right) \\ x_3 &= p_3^{T,12} \\ x &\in P(S) \end{aligned}$$

which is uniquely reached at $x = p^{T,12}$. Since $x \neq p^{T,12}$, player 3 can assure himself strictly more than $f(p^{T,12}) = p_3^{T,13}$. Moreover, $a \left(S, x_i + p_j^{T,i3} \right)$ is the final allocation.

Claim 4: In the subgame that begins when 2 vetoes x , the final payoff for 1 is strictly less than $p_1^{T,13}$.

If T is odd, the final payoff allocation is $p^{T-1,12}$. Hence, 1 gets $p_1^{T-1,12} < p_1^{T,13}$.

If T is even, the final payoff allocation is $p^{T-1,23}$. Hence, 1 gets $p_1^{T-1,23} < p_1^{T,13}$.

Claim 5: In the subgame that begins when 1 chooses $x \in S$, the final payoff is $p^{T,13}$.

Player 3 can assure himself a final payoff of $p_2^{T-1,2} = p_2^{T,13}$ by vetoing. Hence, player 2 gets not less than $p_2^{T,13}$ in any case.

We will prove that 1's optimal proposal is $x = p^{T,12}$.

For any x , if player 2 vetoes, under Claim 4 the final payoff for player 1 is strictly less than $p_1^{T,13}$. Moreover, if player 2 does not veto and $x_3 < p_3^{T,12}$, under Claim 2.1 the final payoff for 3 is $p_3^{T-1,3} = p_3^{T,12}$, irrespective of the value of x . Under Assumption 2(*), if $x_3 < p_3^{T,12}$, player 3 chooses $i = 2$ and hence (again under Claim 2.1) the final payoff is $p^{T-1,13}$, if T is odd, and $p^{T-1,23}$, if T is even. If T is odd, player 1 gets $p_1^{T-1,13} < p_1^{T,13}$. If T is even, 1 gets $p_1^{T-1,23} < p_1^{T,13}$.

Hence, 1 gets less than $p_1^{T,13}$ unless player 2 does not veto and $x_3 \geq p_3^{T,12}$.

Let $x = p^{T,12} - \varepsilon_1$ for some $\varepsilon_1 > 0$. With this proposal, we have

$$\begin{aligned} & \max \left\{ a_3 \left(S, x_1 + p_2^{T,13} \right), a_3 \left(S, x_2 + p_1^{T,23} \right) \right\} \\ &= \max \left\{ a_3 \left(S, p_1^{T,12} - \varepsilon_1 + p_2^{T,13} \right), a_3 \left(S, p_2^{T,12} + p_1^{T,23} \right) \right\} \\ &= \max \left\{ p_3^{T,13} + \varepsilon_3, p_3^{T,23} \right\} = p_3^{T,13} + \varepsilon_3 = a_3 \left(S, x_1 + p_2^{T,13} \right) \end{aligned}$$

and hence $\arg \max_{k \in \{1,2\}} a_3 \left(S, x_k + p_{k^c}^{T,k3} \right) = 1$. Under Claim 3.2, in case 3 does not veto the final payoff is $a \left(S, x_1 + p_2^{T,13} \right)$. In particular 2 gets $p_2^{T,13}$ and hence 2 would not veto x (under Assumption 1). The final payoff for player 1 is $x_1 = p_1^{T,12} - \varepsilon_1$ and hence we can assure that in SP equilibrium 1 gets at least $p_1^{T,12} = p_1^{T,13}$ and, moreover, $x_3 \geq p_3^{T,12}$ and player 2 does not veto.

We will see that, in fact, player 1 cannot get more than $p_1^{T,13}$.

- If $x = p^{T,12}$ then Claim 2.2 implies that 1 can only get $x_1 = p_1^{T,12} = p_1^{T,13}$, or $p_1^{T,32}$ (then 2 receives $x_2 = p_2^{T,12}$), or $a_1 \left(S, x_2 + p_3^{T,12} \right) = a_1 \left(S, p_2^{T,12} + p_3^{T,12} \right) = p_1^{T,12} = p_1^{T,13}$. If 1 gets $p_1^{T,32}$ and 2 gets $p_2^{T,12}$, we study two cases:

- T odd. Then $p_1^{T,23} < p_1^{T,13}$.
- T even. Then $p_2^{T,12} < p_2^{T,13}$ which is not possible.

- If $x \neq p^{T,12}$ then Claim 3.2 implies player 3 receives

$$\max \left\{ a_3 \left(S, x_1 + p_2^{T,13} \right), a_3 \left(S, x_2 + p_1^{T,23} \right) \right\} = f(x) > f(p^{T,12}) = p_3^{T,13}$$

and hence player 1 can get no more than

$$a_1 \left(S, p_3^{T,13} + p_2^{T,13} \right) = p_1^{T,13}.$$

We now prove that there exists at least a SP equilibrium. Consider the following strategy profile:

- At the beginning of the round, player 1 proposes $x = p^{T,12}$.
- After player 1 proposes x , player 2 vetoes iff $x_3 \geq p_3^{T,12}$, $a_3 \left(S, x_1 + p_2^{T,13} \right) < a_3 \left(S, x_2 + p_1^{T,32} \right)$, and $x_2 < p_2^{T,13}$.
- If player 2 vetoes, player 3 chooses B [312] iff $p_3^{T-1,32} \geq p_3^{T-1,12}$.
- If player 2 does not veto, player 3 chooses $i \in \{1, 2\}$ and y following the next rule: If $x_3 < p_3^{T,12}$, he takes $i = 2$ and, afterwards, $y = a \left(S; p_j^{T-1,i3} + p_i^{T-1,i3} - \varepsilon_i \right)$ if T is even, or $y = p^{T-1,3j}$ if T is odd. If $x_3 \geq p_3^{T,12}$, he takes $i = 1$ iff $a_3 \left(S, x_1 + p_2^{T,13} \right) \geq a_3 \left(S, x_2 + p_1^{T,32} \right)$ and, afterwards, $y = a \left(S, x_i + p_j^{T,i3} \right)$.
- After player 3 chooses y and i , player i chooses either x or y following the next rule:
 - If $x_3 < p_3^{T,12}$ and $y_j < p_j^{T,i3}$, he chooses y .
 - If $x_3 < p_3^{T,12}$, $y_j \geq p_j^{T,i3}$, and T is odd, he chooses x iff $y_i < p_i^{T,i3}$.
 - If $x_3 < p_3^{T,12}$, $y_j \geq p_j^{T,i3}$, and T is even, he chooses x iff $y_i < p_i^{T,j3}$.
 - If $x_3 \geq p_3^{T,12}$ and $y_j < p_j^{T,i3}$, and T is odd, he chooses x iff $p_i^{T-1,j3} < x_i$.
 - If $x_3 \geq p_3^{T,12}$ and $y_j < p_j^{T,i3}$, and T is even, he chooses x iff $p_i^{T,j3} < x_i$.
 - If $x_3 \geq p_3^{T,12}$ and $y_j \geq p_j^{T,i3}$, he chooses x iff $y_i < x_i$.
- If player 3 chooses y , then player j vetoes iff $y_j < p_j^{T,i3}$.

- If player j vetoes, then player i chooses $B [i3j]$ iff $p_i^{T-1,ij} \geq p_i^{T-1,3j}$.
- If player 3 chooses x , then j proposes z depending on x_3 , as follows:
 - If $x_3 < p_3^{T,ij}$, then player j proposes $z = x$.
 - If $x_3 \geq p_3^{T,ij}$, then player j proposes $z = a \left(S, x_i + p_3^{T,12} \right)$.
- After player j proposes z , player i chooses either z or x following the next rule:
 - If $x_3 < p_3^{T,ij}$ and $z_3 < p_3^{T,ij}$, he chooses x .
 - If $x_3 < p_3^{T,ij}$, $z_3 \geq p_3^{T,ij}$, and T is odd, he chooses x iff $z_i < p_i^{T-1,i3}$.
 - If $x_3 < p_3^{T,ij}$, $z_3 \geq p_3^{T,ij}$, and T is even, he chooses x iff $z_i < p_i^{T-1,j3}$.
 - If $x_3 \geq p_3^{T,ij}$, $z_3 < p_3^{T,ij}$, and T is odd, he chooses x iff $p_i^{T-1,i3} < x_i$.
 - If $x_3 \geq p_3^{T,ij}$, $z_3 < p_3^{T,ij}$, and T is even, he chooses x iff $p_i^{T-1,j3} < x_i$.
 - If $x_3 \geq p_3^{T,ij}$ and $z_3 \geq p_3^{T,ij}$, he chooses x iff $z_i < x_i$.
- After player i chooses $q \in \{x, z\}$, player 3 vetoes iff $q_3 < p_3^{T,ij}$.
- If player 3 vetoes, then player j chooses $B [ji3]$ iff $p_j^{T-1,j3} \geq p_j^{T-1,i3}$.

In order to prove that this strategy profile constitutes a SP equilibrium with final payoff $p^{T,13}$, we proceed by induction hypothesis on T . For $T = 0$, the result is trivially true. Assume the result is true for less than T rounds. The subgame that arises in the second round of the game with T rounds is strategically equivalent to the game with $T - 1$ rounds. Hence, under the induction hypothesis the strategy profile constitutes a SP equilibrium in the second round and the final payoff is $p^{T-1,\alpha\beta}$, where α and β are the new first proposer and the new pivot, respectively.

Hence, since $p_j^{T-1,j3} < p_j^{T-1,i3}$ iff T is even, a simple backward reasoning shows that the above strategies constitute a SP equilibrium after player j proposes z .

To see that the proposed choice of z is optimal for player j , notice first that player 3 can assure himself at least $p_3^{T-1,3} = p_3^{T,12}$ by vetoing any q . We have two cases:

1. If $x_3 < p_3^{T,12}$, then player i can assure himself at least $p_i^{T-1,j3}$, if T is even, or $p_i^{T-1,i3}$, if T is odd, by choosing x . Hence, the maximum that player j can get is $a_j \left(S; p_i^{T-1,j3} + p_3^{T-1,3} \right) = p_j^{T-1,j3}$, if T is even, or $a_j \left(S; p_i^{T-1,i3} + p_3^{T-1,3} \right) = p_j^{T-1,i3}$, if T is odd. This is what he gets by choosing $z = x$, because it would induce player i to choose $q = x$ and player 3 to veto.
2. If $x_3 \geq p_3^{T,12}$, then player i can assure himself at least x_i by choosing x . Hence, the maximum that player j can get is $a_j \left(S; x_i + p_3^{T,12} \right)$. This is what he gets by choosing $z = a \left(S; x_i + p_3^{T,12} \right)$, because it would induce player i to choose $q = x$ and player 3 not to veto.

Moreover, it is straightforward to check that the final payoff in this subgame (when j chooses z) is given by $a \left(S, x_i + p_3^{T,12} \right)$ when $x_3 \geq p_3^{T,12}$, by $p^{T-1,j3}$ when $x_3 < p_3^{T,12}$ and T is odd, and by $p^{T-1,i3}$ when $x_3 < p_3^{T,12}$ when T is even.

Again, the fact that $p_i^{T-1,i3} < p_i^{T-1,ij}$ iff T is odd and a standard backward reasoning implies that the above strategies constitute a SP equilibrium after player 3 proposes y and i .

We now check that the proposed choice of i and y is optimal for player 3. We have four cases:

Case 1: $x_3 < p_3^{T,12}$ and T is even. In this case, if player 3 does not deviate, $y = a \left(S; p_j^{T-1,i3} + p_i^{T-1,i3} - \varepsilon_i \right)$, player i chooses x and, under Claim 1.1, the final payoff is $p^{T-1,i3}$. Hence, the final payoff for player 3 is $p_3^{T-1,i3} = p_3^{T,12}$. If player 3 deviates so that player i chooses y and player j vetoes, the final payoff is $p^{T-1,12}$. Since $p_3^{T-1,12} < p_3^{T,12}$, player 3 does not improve. If player 3 deviates so that player i chooses y and player j does not veto, y_j must be at least $p_j^{T,i3}$. But player i should get at least $p_i^{T-1,i3}$ because this is

his payoff after choosing x . Hence, player 3 would never get more than $a_3 \left(S; p_j^{T,i3} + p_i^{T-1,i3} \right) = p_3^{T-1,12} < p_3^{T-1,i3}$. Deviation is not profitable.

Case 2: $x_3 < p_3^{T,12}$ and T is odd. In this case, if player 3 does not deviate, $y = p^{T-1,3j}$, player i chooses y , player j does not veto and the final payoff is $y = p^{T-1,3j}$. Hence, the final payoff for player 3 is $p_3^{T-1,3j} = p_3^{T,12}$. If player 3 deviates so that player i chooses x , Claim 1.1 implies that the final payoff is $p^{T-1,3j}$. If player 3 deviates so that player i chooses y and player j does not veto, his maximum final payoff is $a_3 \left(S; p_i^{T-1,3j} + p_j^{T-1,j} \right) = p_3^{T-1,3j}$. In either case, player 3 does not improve.

Case 3: $x_3 \geq p_3^{T,12}$. In this case, if player 3 does not deviate, $y = a \left(S, x_i + p_j^{T,i3} \right)$, player i chooses y , player j does not veto and the final payoff is y . The final payoff for player 3 is $a_3 \left(S; x_i + p_j^{T,i3} \right)$. If player 3 deviates so that player i chooses x , under Claim 1.2 the final payoff for player 3 is $p_3^{T,12}$. The choice of i assures that

$$a_3 \left(S; x_i + p_j^{T,i3} \right) \geq a_3 \left(S; x_j + p_i^{T,j3} \right)$$

but the function $\max \left\{ a_3 \left(S; x'_1 + p_2^{T,13} \right), a_3 \left(S; x'_2 + p_1^{T,32} \right) \right\}$ restricted to $x'_3 \geq p_3^{T,12}$ reaches a minimum when $x' = p^{T,12}$ (lemma), hence $a_3 \left(S; x_i + p_j^{T,i3} \right) \geq p_3^{T,12}$. If player 3 deviates so that player i chooses y and player j vetoes, his final payoff is $p_3^{T-1,12} < p_3^{T,12}$, if T even, or $p_3^{T-1,3j} = p_3^{T,12}$, if T odd. In either case, player 3 does not improve.

We now prove that player 2's strategy is optimal. By vetoing, his final payoff is $p_2^{T-1,2} = p_2^{T,13}$. We distinguish two cases:

Case 1: $x_3 \geq p_3^{T,12}$. We have three subcases:

Subcase 1.1: $a_3 \left(S, x_1 + p_2^{T,13} \right) < a_3 \left(S, x_2 + p_1^{T,32} \right)$. In this subcase, if player 2 does not veto, claim 3.2 implies that his final payoff is x_2 . Hence, it is optimal to veto iff $x_2 < p_2^{T,13}$.

Subcase 1.2: $a_3 \left(S, x_1 + p_2^{T,13} \right) \geq a_3 \left(S, x_2 + p_1^{T,32} \right)$ and $x \neq p_3^{T,12}$. In this subcase, if player 2 does not veto, claim 3.2 implies that his final payoff is $p_2^{T,13}$ and hence it is optimal not to veto.

Subcase 1.3: $x = p_3^{T,12}$. In this subcase, his final payoff is $a_2 \left(S, x_1 + p_2^{T,13} \right) = p_2^{T,13}$ and hence it is optimal not to veto.

Case 2: $x_3 < p_3^{T,12}$. In this case, if player 2 does not veto, $i = 2$ and claim 2.1 implies that his final payoff is $p_2^{T-1,13} > p_2^{T,13}$, if T is odd, or $p_2^{T-1,32} = p_2^{T,13}$, if T is even. Hence, it is optimal not to veto.

We now check that $x = p_3^{T,12}$ is optimal for player 1. If he does not deviate, his final payoff will be $p_1^{T,1}$. Assume he deviates. Player 2 can assure himself $p_2^{T,13}$ by vetoing. Moreover, if player 2 vetoes, player 1 is not better off. Assume then player 2 does not veto. If $x_3 \geq p_3^{T,12}$, claim 3.2 implies that player 3 gets (lemma):

$$\begin{aligned} & \max \left\{ a_3 \left(S; x_1 + p_2^{T,13} \right), a_3 \left(x_2 + p_1^{T,32} \right) \right\} \\ \geq & \max \left\{ a_3 \left(S; p_1^{T,12} + p_2^{T,13} \right), a_3 \left(S; p_2^{T,12} + p_1^{T,32} \right) \right\} \\ = & \max \left\{ p_3^{T,13}, p_3^{T,32} \right\} = p_3^{T,3} \end{aligned}$$

Hence, player 1 gets at most $a_1 \left(S; p_3^{T,3} + p_2^{T,13} \right) = p_1^{T,13}$. Hence, he does not improve.

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