Bargaining without a planner: a non-cooperative approach to the Shapley NTU value*

Juan J. Vidal-Puga
vidalpuga@uvigo.es

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Abstract

We present a simple mechanism of negotiation, based on offers and counteroffers. This leads to a unified solution theory for non-transferable utility (NTU) games that has as special cases the Nash bargaining solution for pure bargaining problems, the Shapley value for transfer utility (TU) games, and the Shapley NTU value for general cooperative games. These results are similar to those of the bargaining mechanism of Hart and Mas-Colell (1996) yielding the consistent value (Maschler and Owen (1898, 1992)). The mechanism presented here solves some problematic issues in Hart and Mas-Colell's model. Furthermore, a natural extension to games with coalition structure, yielding the Owen value (Owen (1977)) for TU games is provided.

Keywords: non-transferable utility games, Shapley NTU value, consistent value, subgame perfect equilibrium

*Work in progress. This is a very preliminary draft. Please don't quote! Latest version at: http://webs.uvigo.es/vidalpuga/.

1 Introduction

Given a group of agents or players that can benefit from cooperation, how will the benefits of this cooperation be distributed among them?

We assume that the players, as rational, completely informed, and selfish[1] utility maximizers negotiate among them. How will they agree on a payoff? Notice that we do not ask what payoff is the fairest, or the most reasonable, but how will they agree on a non-cooperative environment without the presence of a planner.

Hart and Mas-Colell (1996) designed a non-cooperative game such that the consistent value (Maschler and Owen, 1989 and 1992) arises in stationary subgame perfect Nash equilibria. As far as we know, no similar result has been obtained for other extensions of the Shapley value (Shapley (1953)) and the Nash solution (Nash (1950)) to NTU games, such like the Harsanyi value (Harsanyi (1959, 1963)) or the Shapley NTU value[2] (Shapley (1969)).

Even though Hart and Mas-Colell never mention a planner, their model establishes that a player may, eventually, drop out the negotiation table, leaving the rest of the players bargaining among them. The excluded players get zero, with no chance to come back to the negotiation table nor to bargain among themselves.

In this paper, we present an alternative model of bargaining in which players are never excluded.

In Hart and Mas-Colell's model, players should propose payoffs. However, it is not uncommon that offers in real negotiations do not refer to final payoffs, but to general policies, rules or solutions. For example, van Damme (1986) proposes a game in two-player pure bargaining problems in which players' offers are solution concepts (such like the Nash solution, or the Kalai-Smorodinsky solution).

For more than two players, the payoff proposed by a solution concept

[1] i.e. they only care of their own profit. They don't care for the profit of the other players.

[2] A work in this direction has been already taken in Vidal-Puga (2004).

depends on who are the players involved. Following this idea, in our game the offers are rules that assign a different payoff to each subcoalition, i.e. a rule is a function f that assigns to each coalition S a vector $f(S) \in \mathbb{R}^S$ that is feasible for S.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers. Similarly, $\mathbb{R}_{++}$ and $\mathbb{R}_+$ are the set of positive and nonnegative real numbers, respectively. Given any finite set S, we denote the cardinality of S as $|S|$, and the set of all functions from S to $\mathbb{R}$ as $\mathbb{R}^S$. The sets $\mathbb{R}^S_{++}$ and $\mathbb{R}^S_+$ are defined analogously. We also denote the cardinal set of S as 2^S , i.e. $2^S \equiv \{T : T \subset S\}$. A member x of $\mathbb{R}^S$ is an $|S|$-dimensional vector whose coordinates are indexed by members of S; thus, when $i \in S$, we write x_i for $x(i)$. If $x \in \mathbb{R}^S$ (or $x \in \mathbb{R}^N$) and $T \subset S$ (or $T \subset N$), we write x_T for the restriction of x to T, i.e. the members of $\mathbb{R}^T$ whose ith coordinate is x_i. With some abuse of notation, given $x \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we write $(x, a) \in \mathbb{R}^{S\cup\{i\}}$ for the member of $\mathbb{R}^{S\cup\{i\}}$ whose ith coordinate is a and whose restriction to S is x. Given $x, y \in \mathbb{R}^S$, we write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$.

Let $N = \{1, 2, ..., n\}$ be a finite set of *players*. Nonempty subsets of N are called *coalitions*. A *non-transferable utility (NTU) form*[3] on N is a correspondence V that assigns to each coalition S a subset $V(S) \subset \mathbb{R}^S$ satisfying the following properties:

(A1) For each $S \subset N$, the set $V(S)$ is nonempty, closed, convex, *comprehensive* (i.e., if $x \in V(S)$ and $y \leq x$, then $y \in V(S)$), and *bounded above* (i.e., for each $x \in \mathbb{R}^S$, the set $\{y \in V(S) : y \geq x\}$ is compact).

(A2) *Normalization*: For each $i \in N$, the maximum of $\{x : x \in V(\{i\})\}$, which we denote as ω_i, is nonnegative.

[3] As in Hart and Mas-Colell (1996), we use the term *form* instead of the most usual *game* in order to avoid confusions with non-cooperative games.

(A3) *Zero-Monotonicity*: For each $S \subset N$, $x \in V(S)$ and $i \notin S$, we have $(x, \omega_i) \in V(S \cup \{i\})$. In particular, this implies that $(\omega_i)_{i \in S} \in \mathbb{R}^S$ belongs to $V(S)$.

(A4) The boundary of $V(N)$, which we denote as $\partial V(N)$, is smooth (i.e., at any point of $\partial V(N)$ there exists a unique outward normal direction) and nonlevel (i.e. the outward normal vector at any point of $\partial V(N)$ has all its coordinates positive.)

(A5) For any outward normal vector λ at any point of $\partial V(N)$, there exists $\max_{x \in V(S)} \left\{\sum_{i \in S} \lambda_i x_i\right\}$ for all $S \subset N$.

Properties (A1), (A2), (A3), and (A4) are standard properties. The normalization given in (A2) does not affect our results. Property (A4) has been previously used by Hart and Mas-Colell (1996, in hypothesis (A2), page 359) and Serrano (1997, in assumption A4, page 61). The hypothesis in Hart and Mas-Colell (1996) is stronger, since it requires smoothness and nonlevelness in every coalition $S \subset N$.

Property (A5) deserves further discussion. It means that, if we project a supporting hyperplane of $\partial V(N)$ into $\mathbb{R}^S$, we get a supporting hyperplane of $\partial V(S)$ (up to a linear translation). This property is satisfied by all transfer utility forms and pure bargaining problems (see the formal definitions below), and it is easily implied by property (A5) in Vidal-Puga (2004). If fact, Property (A5) is only violated in degenerate situations in which the projection of $\partial V(N)$ is flat and $\partial V(S)$ asymptotically approaches it (see Figure 1.)

In Section 9, we see how this property can be avoided.

A *pure bargaining problem* on N is a pair (D, d) where $D \subset \mathbb{R}^N$ and $d \in D$. The set D is assumed to be closed, convex, comprehensive, bounded above, and its frontier ∂D is smooth and nonlevel. Furthermore, $d \in D \cap \mathbb{R}^N_+$. A pure bargaining problem (D, d) may then be expressed as the following NTU game on N:

$$V(S) = \left\{x \in \mathbb{R}^S : x \leq d\right\}$$

for all $S \subsetneq N$, and $V(N) = D$.

A *transfer utility (TU) form* on N is a function $v : 2^S \to \mathbb{R}$ that assigns to each coalition S a real number $v(S)$ with $v(S \setminus \{i\}) \leq v(S)$ for all $i \in S \subset N$, and $v(\emptyset) = 0$. A TU form v on N may also be expressed as the following NTU game on N:

$$V(S) = \left\{ x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S) \right\} \tag{1}$$

for all $S \subset N$.

Let Π^N be the set of all permutations of players in N. Given $S \subset N$, let Π^S denote the set of orders of the players in S. For notational convenience, we write $\Pi^\emptyset \equiv \{\emptyset\}$. Given $\pi \in \Pi^N$ and $i \in N$, we define P_i^π as the set of players who come before i in the order π, namely

$$P_i^\pi \equiv \{j \in N : \pi(j) < \pi(i)\}.$$

Let v be a TU game on N and let $\pi \in \Pi^N$. Given $i \in N$, we define the *marginal contribution* of player i under the order π in the game v as

$$m_i(\pi, v) \equiv v(P_i^\pi \cup \{i\}) - v(P_i^\pi) \in \mathbb{R}.$$

The *Shapley value* (Shapley (1953)) of a TU form v on N is the vector $Sh(N, v) \in \mathbb{R}^N$ whose ith coordinate is given by

$$Sh_i(N, v) \equiv \frac{1}{|\Pi^N|} \sum_{\pi \in \Pi^N} m_i(\pi, v) \in \mathbb{R}.$$

Let $\lambda \in \mathbb{R}^N_{++}$ and let $S \subset N$. We define

$$v^\lambda(S) \equiv \sup \left\{ \sum_{i \in S} \lambda_i x_i : x \in V(S) \right\}.$$

A point $x \in V(N)$ is a *Shapley NTU value* (Shapley (1969)) payoff of V if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that $\lambda_i x_i = Sh_i\left(N, v^\lambda\right)$ for all $i \in N$.

Players will decide how to share the benefit of their mutual collaboration, and their only threat is to refuse to collaborate. Their payoff will only depend on the identity of the players who actually collaborate. Thus, we define a *rule* as a function f which assigns to each coalition S a vector $f(S) \in V(S)$. Formally, a rule is a "payoff configuration" (see Hart and Mas-Colell (1996)). However, a rule should not be interpreted as a payoff for every coalition, but as an index that indicates the payoff when a particular coalition of players has agreed to collaborate. We denote the set of all rules as $\mathcal{F}$.

3 The model

Assume coalition S should decide an offer. If $S = \{i\}$, then player i proposes a rule and we say that coalition S has decided this offer with consensus.

Suppose we know how a coalition of size $|S| - 1$ decide an offer. This offer can be decided with or without consensus. We now describe how coalition S does. First, a player i is randomly chosen out of S, being each player equally likely to be chosen. The coalition $S \backslash \{i\}$ decides an offer f (we know how, by induction hypothesis). If this proposed offer is reached with consensus, then player i can either agree or disagree with f. If he agrees, then f will be the offer of coalition S with consensus.

In case he disagrees, or the offer f is reached without consensus, player i can make a counteroffer g. If all the players in $S \backslash \{i\}$ accept (they are asked in some prespecified order) then g will be the offer of coalition S with consensus. If at least a player in $S \backslash \{i\}$ rejects, then with probability ρ a new player is randomly chosen out of S, and the process repeats. With probability $1 - \rho$, there is breakdown and the offer of coalition S will be given by $f^{-i}(T) = f(T)$ if $i \notin T$ and $f^{-i}(T) = (f(T \backslash \{i\}), \omega_i)$ if $i \in T$. In the latter case, the offer is reached without consensus.

In the case the coalition is N, then the proposed rule g (that can reached with or without consensus) is implemented, i.e. each player i receives $g_i(N)$.

We work with stationary strategies. A strategy is stationary if it only depends on the elements present on the negotiation table, and not on the history that yields these elements.

From now on, when we say equilibrium, we mean stationary subgame perfect Nash equilibrium.

It is useful to describe the game for two players, say 1 and 2. In this case, we are in a pure bargaining problem. First a player (say, player 1) is chosen and makes an offer[4]. If player 2 agrees, this offer is implemented. If player 2 disagrees, then he makes a counteroffer. If player 1 accepts the counteroffer, it is implemented. If player 1 rejects, then with probability ρ the process is repeated with a new random proposer. With probability $1 - \rho$, the game finishes in disagreement, and each player gets his individually rational payoff.

Thus, the game for two players combines the alternate-proposer and the random-proposer versions of the Rubinstein's bargaining model, both of them yielding the Nash bargaining solution as ρ approaches 1. However, the risk of breakdown only arises after the counteroffer, which makes the first offer stage innocuous.

As a conclusion, the game for two players is (essentially) the same as the random-proposer version of Rubinstein's model. This analysis can be extended for more than two players when the game is a pure bargaining problem. Thus, we have (cf. Theorem 3 in Hart and Mas-Colell, 1996) the following result:

Theorem 3.1 *For a pure bargaining problem, for each ρ there is at least one equilibrium. Moreover, any equilibrium payoff converge to the Nash bargaining solution as ρ approaches* 1*.*

For more than two players, the offer of the coalition is not innocuous, as it allows a reassignment of utilities among those that leave the player indifferent. See Example 5 in Vidal-Puga (2004).

For TU games, we have the following result (that is consequence of Theorem 6.2 Section 6, cf. Theorem 2 in Hart and Mas-Colell):

Theorem 3.2 *Let (N, v) be a TU form. Then, for each $0 \leq \rho < 1$ there is a unique expected equilibrium payoff, which coincides with the Shapley*

[4] This offer is obviously made with consensus.

value. Moreover, as ρ approaches 1, *the final equilibrium payoffs approach the Shapley value.*

4 Analysis of the subgames

Let π_1 be the first player in being chosen, i.e. coalition $N\backslash \{\pi_1\}$ must make an offer. In order to make an offer, a second player is randomly chosen out of $N\backslash \{\pi_1\}$. Let π_2 be this player, and so on. Each order $\pi = [\pi_1, ..., \pi_n] \in \Pi^N$ is equally likely to be chosen. Player π_n is the one who makes the first offer. Once an offer is made, we go back from π_n to π_{n-1}, from π_{n-1} to π_{n-2}, and so on.

We can distinguish two different kinds of subgames (see Figure 4b):

1. Subgame that begins when coalition S is due to make an offer. This subgame is characterized by an order $\pi_{N\backslash S} = [\pi_1, \pi_2, ..., \pi_k] \in \Pi^{N\backslash S}$. Once a player $i \in S$ is randomly chosen out of S, we begin the subgame in which coalition $S\backslash \{i\}$ is due to make an offer. This new subgame is characterized by $\pi^{+i}_{N\backslash S} \in \Pi^{(N\backslash S)\cup\{i\}}$ where $\pi^{+i}_t = \pi_t$ for all $t = 1, ..., k$ and $\pi^{+i}_{k+1} = i$, and so on. We denote these subgames as $A\left(S, \pi_{N\backslash S}\right)$, $A\left(S\backslash \{i\}, \pi^{+i}_{N\backslash S}\right)$, and so on.

2. Subgame that begins when coalition S makes an offer f to player i. This subgame is characterized by the rule f and an order $\pi_{N\backslash S} = [\pi_1, \pi_2, ..., \pi_k] \in \Pi^{N\backslash S}$ with $i = \pi_k$. If player i agrees, we move to the subgame in which coalition $S \cup \{i\}$ makes the offer f to player π_{k-1}. In the particular case $S = N$, $f(N)$ is implemented.
   We denote these subgames as $B\left(S, f, \pi_{N\backslash S}\right)$ and $B\left(S \cup \{i\}, f, \pi^{-i}_{N\backslash S}\right)$, respectively, where $\pi^{-i}_{N\backslash S} = [\pi_1, \pi_2, ..., \pi_{k-1}] \in \Pi^{(N\backslash S)\backslash\{i\}}$. Note that in $\pi^{+i}_{N\backslash S}$ we have $i \in S$ and we *add* player i to the order $\pi_{N\backslash S}$, whereas in $\pi^{-i}_{N\backslash S}$ we have $i \in N\backslash S$ and we *remove* player i out of the order $\pi_{N\backslash S}$.

Hence, we have the following transitions between subgames (see Figure

4b):

$$
\begin{aligned}
A\left(S,\pi_{N\backslash S}\right) &\rightarrow \begin{cases} A\left(S\backslash\{i\},\pi^{+i}_{N\backslash S}\right) & \text{if } S\neq\{i\} \\ B\left(S,f,\pi_{N\backslash S}\right) & \text{if } S=\{i\} \end{cases} \\
B\left(S,f,\pi_{N\backslash S}\right) &\rightarrow \begin{cases} \left.\begin{array}{cc} B\left(S\cup\{i\},f,\pi^{-i}_{N\backslash S}\right) & \text{or} \\ B\left(S\cup\{i\},f^{-i},\pi^{-i}_{N\backslash S}\right) & \text{or} \\ A\left(S\cup\{i\},\pi^{-i}_{N\backslash S}\right) & \end{array}\right\} & \text{if } S\neq N \\ f(N) & \text{if } S=N. \end{cases}
\end{aligned}
$$

The game is $A(N,\emptyset)$. For notational convenience, we can state that $A(\emptyset,\pi)=B(\emptyset,\pi,w)$, where $\emptyset$ has "agreed" without consensus on the rule w with $w(S)=\omega_S$ for all $S\subset N$.

5 Characterization of equilibria

We denote the equilibrium payoff in the subgame $A(S,[\pi_1...\pi_k])$ as $a(S,[\pi_1...\pi_k])\in\mathbb{R}^N$.

Given $\pi\in\Pi^N$ and $i\in N$, the set of predecessors of i under π is

$$P^{\pi}_{N,i}\equiv\{j\in N:\pi(j)<\pi(i)\}$$

and the set of successors of i under π is

$$S^{\pi}_{N,i}\equiv\{j\in N:\pi(j)>\pi(i)\}.$$

Proposition 5.1 *Given $S\subset N$ and $\pi_{N\backslash S}\in\Pi^{N\backslash S}$, we define $a\left(S,\pi_{N\backslash S}\right)\in\mathbb{R}^{N\backslash S}$ as follows:*

$$
\begin{aligned}
a(S,[\pi_1...\pi_k]) &= \frac{1}{|S|}\sum_{j\in S}a(S\backslash\{j\},[\pi_1...\pi_k\ j]) \text{ for all } k=0,...,n-1 \\
a(\emptyset,\pi) &\in \partial V(N) \\
a_i(\emptyset,\pi) &= \min_{y\in V\left(S^{\pi}_{N,i}\right)}\ \max_{\left(a_{P^{\pi}_i}(\emptyset,\pi),x_i,z^k(i,y)\right)\in V(N)} x_i \text{ for all } i=\pi_k\in N
\end{aligned}
$$

where $z^k(i,y) \in \mathbb{R}^{S^\pi_{N,i}}$ *is inductively defined as follows:*

$$\begin{aligned} z^t(i,y) &= \rho a_{S^\pi_{N,i}}(\{\pi_t,...,\pi_n\},[\pi_1...\pi_{t-1}]) + (1-\rho)z^{t-1}(i,y) \ \textit{for all } t=1,...,k \\ z^0(i,y) &= y. \end{aligned}$$

When all the offers are accepted, these $a(S,\pi_{N\setminus S})$ *characterize the equilibrium payoffs in each subgame* $A(S,\pi_{N\setminus S})$*.*

Proof. (Sketch) Given $S \subset N$ and $\pi_{N\setminus S} \in \Pi^{N\setminus S}$, let $b(S,\pi_{N\setminus S})$ denote the final payoff in the subgame $A(S,[\pi_1...\pi_k])$. We will prove that $a(S,\pi_{N\setminus S}) = b(S,\pi_{N\setminus S})$. Since we move from $A(S,[\pi_1...\pi_k])$ to $A(S\setminus\{\pi_{k+1}\},[\pi_1...\pi_{k+1}])$ by randomly choosing player π_{k+1} out of S, it is clear that the equilibrium payoffs satisfy

$$b(S,[\pi_1...\pi_k]) = \frac{1}{|S|}\sum_{j\in S} b(S\setminus\{j\},[\pi_1...\pi_k\ j])$$

for all $k=0,...,n-1$.

Assume player i is the first one to be chosen (i.e. $\pi_1 = i$ and $S^\pi_{N,i} = N\setminus\{i\}$). Players in $N\setminus\{i\}$ will propose a rule f with or without consensus. We will compute the minimum payoff player i can assure for himself. This can be done by rejecting[5] f and proposing a rule that leaves players in $N\setminus\{i\}$ indiferent between voting 'yes' or 'no', i.e. player i would propose a rule g satisfying

$$g_j(N) = \rho b_j(N,\emptyset) + (1-\rho) f_j(N\setminus\{i\}) \equiv \tilde{z}^1_j(i,f(N\setminus\{i\}))$$

for all $j \in S^\pi_{N,i}$. This means that player i can assure himself a payoff of at least

$$\max_{(x_i,\tilde{z}^1(i,f(N\setminus\{i\})))\in V(N)} x_i.$$

For simplicity, we assume that, in case of indiference between accepting or rejecting an offer, player i strictly prefers to accept[6]. Hence, player i would

[5]only in case f is a proposal reached by consensus.

[6]A more elaborate argument should be developed o explain the case in which this tie-breaking rule does not hold.

accept any rule f satisfying the following two conditions:

$$\begin{aligned} f_i(N) &= \min_{y \in V\left(S^{\pi}_{N,i}\right)} \max_{(x_i, \tilde{z}^1(i,y)) \in V(N)} x_i = b_i(N \setminus \{i\}, [i]) \\ f\left(S^{\pi}_{N,i}\right) &\in \arg \min_{y \in V\left(S^{\pi}_{N,i}\right)} \max_{(x_i, \tilde{z}^1(i,y)) \in V(N)} x_i. \end{aligned}$$

A similar argument can be done for player i when he is the second one in the order. This means that there is an additional player, say π_1, waiting for a proposal and player i receives a proposal f from players in $N \setminus \{\pi_1, i\}$. Any proposal f is bound to be accepted by player π_1 as long as

$$\begin{aligned} f_{\pi_1}(N) &= b_{\pi_1}(N \setminus \{\pi_1\}, [\pi_1]) \\ f(N \setminus \{\pi_1\}) &\in \arg \min_{y \in V\left(S^{\pi}_{N,\pi_1}\right)} \max_{\left(x_{\pi_1}, \tilde{z}^1(\pi_1, y)\right) \in V(N)} x_{\pi_1}. \end{aligned}$$

Taking these constrains into account, player i can reject[7] f and propose an acceptable rule (acceptable for player π_1) that leaves players in $N \setminus \{\pi_1, i\}$ indiferent between voting ‘yes’ or ‘no’, i.e. player i can propose a rule g satisfying the above conditions and furthermore

$$\begin{aligned} g_j(N) &= \rho b_j(N \setminus \{\pi_1\}, [\pi_1]) + (1-\rho)\, \tilde{z}^1_j(i, f(N \setminus \{\pi_1, i\})) \\ &\equiv \tilde{z}^2_j(i, f(N \setminus \{\pi_1, i\})) \end{aligned}$$

for all $j \in N \setminus \{\pi_1, i\}$. Notice that $\tilde{z}^1(i, f(N \setminus \{\pi_1, i\}))$ is the payoff that the members of $N \setminus \{\pi_1, i\}$ would get if they vote ‘no’ and there is a partial breakdown (hence player π_1 makes an offer that leaves them indifferent between voting ‘yes’ or ‘no’).

This means that player i can assure himself a payoff of at least

$$\max_{\left(b_{\pi_1}(N \setminus \{\pi_1\}, [\pi_1]), x_i, \tilde{z}^2(i, f(N \setminus \{\pi_1, i\}))\right) \in V(N)} x_i.$$

Hence, player i would accept any rule satisfying the following two condi-

[7]only in case f is a proposal reached by consensus.

tions:

$$\begin{aligned} f_i(N) &= \min_{y\in V\left(S^{\pi}_{N,i}\right)} \max_{\left(b_{\pi_1}(N\setminus\{\pi_1\},[\pi_1]),x_i,\tilde{z}^2(i,y)\right)\in V(N)} x_i \\ &= b_i\left(N\setminus\{\pi_1,i\},[\pi_1 i]\right) \\ f\left(S^{\pi}_{N,i}\right) &\in \arg\min_{y\in V\left(S^{\pi}_{N,i}\right)} \max_{\left(b_{\pi_1}(N\setminus\{\pi_1\},[\pi_1]),x_i,\tilde{z}^2(i,y)\right)\in V(N)} x_i. \end{aligned}$$

Once the turn reaches the last player, say π_n, the best acceptable offer is to propose f such that $f(N) = a(\emptyset,\pi) \in \partial V(N)$ and

$$\begin{aligned} f_i(N) &= b_i\left(S^{\pi}_{N,i}, \pi_{N\setminus S^{\pi}_{N,i}}\right) \\ f\left(S^{\pi}_{N,i}\right) &\in \arg\min_{y\in V\left(S^{\pi}_{N,i}\right)} \max_{\left(b_{P^{\pi}_{N,i}}(N\setminus\{\pi_1\},[\pi_1]),x_i,\tilde{z}^k(i,y)\right)\in V(N)} x_i \end{aligned}$$

for all $i = \pi_k \in N$. Hence, the values $b\left(S, \pi_{N\setminus S}\right)$ and $\tilde{z}(i,y)$ coincide with $a\left(S,\pi_{N\setminus S}\right)$ and $z(i,y)$ respectively, as desired. ■

Assume $\partial V(N)$ is flat, i.e.

$$V(N) = \left\{x \in \mathbb{R}^N : \sum_{i\in N} \lambda_i x_i \le v^{\lambda}(N)\right\}. \tag{2}$$

Thus, we can restate Proposition 5.1 as follows:

Proposition 5.2 *If $\partial V(N)$ is flat, the equilibrium payoffs are characterized by*

$$a(S,[\pi_1...\pi_k]) = \frac{1}{|S|}\sum_{j\in S} a([\pi_1...\pi_k j]) \text{ for all } k = 0,...,n-1 \tag{3}$$

$$\sum_{i\in N} \lambda_i a_i(\emptyset,\pi) = v^{\lambda}(N) \tag{4}$$

$$\begin{aligned} \lambda_i a_i(\emptyset,\pi) &= v^{\lambda}(N) - \sum_{j\in P^{\pi}_{N,i}} \lambda_j a_j(\emptyset,\pi) \\ &\quad - \sum_{l=1}^{k} \rho(1-\rho)^{l-1} \sum_{j\in S^{\pi}_{N,i}} \lambda_j a_j\left(\{\pi_{k-l+1},...,\pi_n\},[\pi_1...\pi_{k-l}]\right) \\ &\quad - (1-\rho)^k v^{\lambda}\left(S^{\pi}_{N,i}\right) \end{aligned} \tag{5}$$

for all $i \in N$.

Proof. Conditions (3) and (4) are clear. We prove condition (5). Under (2),

$$\max_{\left(a_{P_i^\pi}(\emptyset,\pi),x_i,z^k(i,y)\right)\in V(N)} x_i = \frac{1}{\lambda_i}\left[v^\lambda(N) - \sum_{j\in P_{N,i}^\pi}\lambda_j a_j(\emptyset,\pi) - \sum_{j\in S_{N,i}^\pi}\lambda_j z_j^k(i,y)\right]$$

and so

$$\begin{aligned}\lambda_i a_i(\emptyset,\pi) &= \min_{y\in V\left(S_{N,i}^\pi\right)}\left[v^\lambda(N) - \sum_{j\in P_{N,i}^\pi}\lambda_j a_j(\emptyset,\pi) - \sum_{j\in S_{N,i}^\pi}\lambda_j z_j^k(i,y)\right] \\ &= v^\lambda(N) - \sum_{j\in P_{N,i}^\pi}\lambda_j a_j(\emptyset,\pi) - \max_{y\in V\left(S_{N,i}^\pi\right)}\sum_{j\in S_{N,i}^\pi}\lambda_j z_j^k(i,y).\end{aligned}$$

We compute this last term. We have to prove

$$\begin{aligned}\max_{y\in V\left(S_{N,i}^\pi\right)}\sum_{j\in S_{N,i}^\pi}\lambda_j z_j^k(i,y) &= \sum_{l=1}^{k}\rho(1-\rho)^{l-1}\sum_{j\in S_{N,i}^\pi}\lambda_j a_j(\{\pi_{k-l+1}...\pi_n\},[\pi_1...\pi_{k-l}]) \\ &\quad+(1-\rho)^k v^\lambda\left(S_{N,i}^\pi\right) \qquad (6)\end{aligned}$$

By definition,

$$\max_{y\in V\left(S_{N,i}^\pi\right)}\sum_{j\in S_{N,i}^\pi}\lambda_j z_j^0(i,y) = \max_{y\in V\left(S_{N,i}^\pi\right)}\sum_{j\in S_{N,i}^\pi}\lambda_j y_j = v^\lambda\left(S_{N,i}^\pi\right)$$

and hence (6) is satisfied for $k=0$, i.e. $i=\pi_1$. Assume (6) is satisfied for $k-1$, i.e.

$$\begin{aligned}\max_{y\in V\left(S_{N,i}^\pi\right)}\sum_{j\in S_{N,i}^\pi}\lambda_j z_j^{k-1}(i,y) &= \sum_{l=1}^{k-1}\rho(1-\rho)^{l-1}\sum_{j\in S_{N,i}^\pi}\lambda_j a_j(\{\pi_{k-l}...\pi_n\},[\pi_1...\pi_{k-l-1}]) \\ &\quad+(1-\rho)^{k-1} v^\lambda\left(S_{N,i}^\pi\right)\end{aligned}$$

By definition of $z^k(i,y)$ and the induction hypothesis,

$$\begin{aligned}
\max_{y\in V\left(S^{\pi}_{N,i}\right)} \sum_{j\in S^{\pi}_{N,i}} \lambda_j z^k_j(i,y) &= \rho \sum_{j\in S^{\pi}_{N,i}} \lambda_j a_j\left(\{\pi_k...\pi_n\},[\pi_1...\pi_{k-1}]\right) \\
&\quad + (1-\rho) \max_{y\in V\left(S^{\pi}_{N,i}\right)} \sum_{j\in S^{\pi}_{N,i}} \lambda_j z^{k-1}_j(i,y) \\
&= \rho \sum_{j\in S^{\pi}_{N,i}} \lambda_j a_j\left(\{\pi_k...\pi_n\},[\pi_1...\pi_{k-1}]\right) \\
&\quad + \sum_{l=2}^{k} \rho(1-\rho)^{l-1} \sum_{j\in S^{\pi}_{N,i}} \lambda_j a_j\left(\{\pi_{k-l+1}...\pi_n\},[\pi_1...\pi_{k-l}]\right) \\
&\quad + (1-\rho)^k v^{\lambda}\left(S^{\pi}_{N,i}\right)
\end{aligned}$$

which coincides with (6). ■

We prove that these equations characterize the Shapley NTU value.

Lemma 5.1 *If $\partial V(N)$ is flat, then $a(S,[\pi_1...\pi_k]) \in \partial V(N)$ for all $k = 0,...,n$ and $S = \{\pi_{k+1},...,\pi_n\}$.*

Proof. We proceed by backwards induction on k. For $k=n$, the result is given by (4). Assume the result is true for $k+1$. Then, by (3)

$$a(S,[\pi_1...\pi_k]) = \frac{1}{|S|}\sum_{j\in S} a(S\backslash\{j\},[\pi_1...\pi_k j]).$$

By induction hypothesis, each $a([\pi_1...\pi_k j])$ belongs to $\partial V(N)$. Thus,

$$\begin{aligned}
\sum_{i\in N} \lambda_i a_i(\pi_1...\pi_k) &= \sum_{i\in N} \lambda_i \frac{1}{|S|}\sum_{j\in S} a_i(S,[\pi_1...\pi_k j]) \\
&= \frac{1}{|S|}\sum_{j\in S}\sum_{i\in N} \lambda_i a_i(S,[\pi_1...\pi_k j]) \\
&= \frac{1}{|S|}\sum_{j\in S} v^{\lambda}(N) \\
&= v^{\lambda}(N).
\end{aligned}$$

■

Lemma 5.2 *Let $i \in N$ and $\pi \in \Pi^N$ such that $\pi(i) = 1$ (i.e. $\pi_1 = i$). Then $a_i(\emptyset, \pi) = a_i(N \backslash \{i\}, [i])$.*

Proof. Let $\pi \in \Pi$ such that $\pi(i) = 1$, i.e. $\pi = [i\pi_2 ... \pi_n]$. By (5), $\lambda_i a_i(\pi)$ depends on $V\left(S^{\pi}_{N,i}\right)$ but not on the order $[\pi_2 .... \pi_n]$. Thus, $\lambda_i a_i(\pi') = \lambda_i a_i(\pi)$ for all $\pi' \in \Pi^N$ with $\pi'(i) = 1$. By (3) and a standard backwards induction argument, it is not difficult to show that $\lambda_i a_i(\emptyset, \pi) = \lambda_i a_i(N \backslash \{i\}, [i])$. ■

Lemma 5.3 *If $\partial V(N)$ is flat,*

$$\lambda_i \left[a_i(N \backslash \{i\}, [i]) - \rho a_i(N, \emptyset)\right] = (1 - \rho)\left[v^{\lambda}(N) - v^{\lambda}(N \backslash \{i\})\right]$$

for all $i \in N$.

Proof. By (5) and Lemma 5.2,

$$\begin{aligned} \lambda_i a_i(N \backslash \{i\}, [i]) &= v^{\lambda}(N) - \rho \sum_{j \in S^{\pi}_{N,i}} \lambda_j a_j(N, \emptyset) - (1 - \rho) v^{\lambda}\left(S^{\pi}_{N,i}\right) \\ &= (1 - \rho) v^{\lambda}(N) + \rho \left(v^{\lambda}(N) - \sum_{j \in N \backslash \{i\}} \lambda_j a_j(N, \emptyset) \right) - (1 - \rho) v^{\lambda}(N \backslash \{i\}) \end{aligned}$$

by (4),

$$\lambda_i a_i(N \backslash \{i\}, [i]) = (1 - \rho) v^{\lambda}(N) + \rho \lambda_i a_i(N, \emptyset) - (1 - \rho) v^{\lambda}(N \backslash \{i\})$$

from where the result is easily deduced. ■

Proposition 5.3 *If $\partial V(N)$ is flat, the unique equilibrium payoff when all the offers are accepted is the Shapley NTU value.*

Proof. We proceed by induction on the number of players. For $n = 1$,

$$\lambda_1 a_1(N, \emptyset) = \lambda_1 a_1(N \backslash \{i\}, [i]) = v^{\lambda}(N) = Sh_i\left(N, v^{\lambda}\right).$$

Assume the result is true for $n - 1$. Fix $\alpha \in N$. Let $N^* = N \backslash \{\alpha\}$. Given $\pi^* \in \Pi^{N^*}$, we define $\pi = [\alpha \pi^*_1 ... \pi^*_n]$ and

$$b(S, [\pi^*_1 ... \pi^*_k]) \equiv a(S, [\alpha \pi^*_1 ... \pi^*_k])$$

where $S = \left\{\pi^*_{k+1}, ..., \pi^*_n\right\}$.

It is not difficult to see that $S = S^{\pi}_{N,i} = S^{\pi^*}_{N^*,i}$ and $P^{\pi}_{N,i} = P^{\pi^*}_{N^*,i} \cup \{\alpha\}$ for all $i \in N^*$.

We prove that these points satisfy (3), (4) and (5). Given $k < n-1$, let $i = \pi^*_k$. Thus,

$$\begin{aligned}
b\left(S, [\pi^*_1...\pi^*_k]\right) &= a\left(S, [\alpha\pi^*_1...\pi^*_k]\right) \\
&= \frac{1}{\left|S^{\pi}_{N,i}\right|} \sum_{j \in S^{\pi}_{N,i}} a\left(S\backslash\{j\}, [\alpha\pi^*_1...\pi^*_k j]\right) \\
&= \frac{1}{\left|S^{\pi^*}_{N^*,i}\right|} \sum_{j \in S^{\pi^*}_{N^*,i}} b\left(S\backslash\{j\}, [\pi^*_1...\pi^*_k j]\right).
\end{aligned}$$

Hence, (3) is satisfied.

We now check (5) is satisfied. Let $a\left(N, [\alpha\pi_{-1}]\right) = a\left(N, \emptyset\right)$. Then,

$$\begin{aligned}
\lambda_i b_i\left(\emptyset, \pi^*\right) &= \lambda_i a_i\left(\emptyset, \pi\right) \\
&= v^{\lambda}(N) - \sum_{j \in P^{\pi}_{N,i}} \lambda_j a_j\left(\emptyset, \pi\right) \\
&\quad - \sum_{l=1}^{k+1} \rho\left(1-\rho\right)^{l-1} \sum_{j \in S^{\pi}_{N,i}} \lambda_j a_j\left([\alpha\pi_1...\pi_{k-l}]\right) \\
&\quad - \left(1-\rho\right)^{k+1} v^{\lambda}\left(S^{\pi}_{N,i}\right) \\
&= v^{\lambda}(N) - \lambda_{\alpha} a_{\alpha}(\pi) - \sum_{j \in P^{\pi^*}_{N^*,i}} \lambda_j a_j(\pi) \\
&\quad - \sum_{l=1}^{k} \rho\left(1-\rho\right)^{l-1} \sum_{j \in S^{\pi^*}_{N^*,i}} \lambda_j a_j\left([\alpha\pi_1...\pi_{k-l}]\right) \\
&\quad - \rho\left(1-\rho\right)^{k} \sum_{j \in S^{\pi^*}_{N^*,i}} \lambda_j a_j(\emptyset) - \left(1-\rho\right)^{k+1} v^{\lambda}\left(S^{\pi^*}_{N^*,i}\right). \qquad (7)
\end{aligned}$$

Let $\left(N^*, v^{\lambda,\alpha}\right)$ be a TU form defined as follows:

$$v^{\lambda,\alpha}(T) \equiv \rho \sum_{j \in T} \lambda_j a_j(\emptyset) + (1-\rho)\, v^{\lambda}(T)$$

for all $T \subset N^*$. In particular, by Lemma 5.1,

$$\begin{aligned} v^{\lambda,\alpha}(N^*) &= \rho \sum_{j \in N^*} \lambda_j a_j(\emptyset) + (1-\rho) v^{\lambda}(N^*) \\ &= \rho \left(v^{\lambda}(N) - \lambda_\alpha a_\alpha(\emptyset) \right) + (1-\rho) v^{\lambda}(N \setminus \{\alpha\}) \end{aligned}$$

by applying Lemma 5.2 and Lemma 5.3,

$$\begin{aligned} v^{\lambda,\alpha}(N^*) &= v^{\lambda}(N) - \lambda_\alpha a_\alpha(\pi) + \lambda_\alpha \left[a_\alpha(\pi) - \rho a_\alpha(\emptyset) \right] \\ &\quad - (1-\rho) v^{\lambda}(N) + (1-\rho) v^{\lambda}(N \setminus \{\alpha\}) \\ &= v^{\lambda}(N) - \lambda_\alpha a_\alpha(\pi) + (1-\rho) \left[v^{\lambda}(N) - v^{\lambda}(N \setminus \{\alpha\}) \right] \\ &\quad - (1-\rho) v^{\lambda}(N) + (1-\rho) v^{\lambda}(N \setminus \{\alpha\}) \\ &= v^{\lambda}(N) - \lambda_\alpha a_\alpha(\pi). \end{aligned} \tag{8}$$

Then, (7) can be restated as follows:

$$\begin{aligned} \lambda_i b_i(\pi^*) &= v^{\lambda,\alpha}(N^*) - \sum_{j \in P^{\pi^*}_{N^*,i}} \lambda_j a_j(\pi) \\ &\quad - \sum_{l=1}^{k} \rho (1-\rho)^{l-1} \sum_{j \in S^{\pi^*}_{N^*,i}} \lambda_j b_j \left(\left[\pi^*_1 ... \pi^*_{k-l} \right] \right) \\ &\quad - (1-\rho)^k v^{\lambda,\alpha} \left(S^{\pi^*}_{N^*,i} \right). \end{aligned}$$

and thus these vectors satisfy (5) for $v^{\lambda,\alpha}$. We prove now that they also satisfy (4) for $v^{\lambda,\alpha}$. By Lemma 5.1 and (8),

$$\sum_{i \in N^*} b_i(\pi^*) = \sum_{i \in N \setminus \{\alpha\}} a_i(\pi) = v^{\lambda}(N) - \lambda_\alpha a_\alpha(\pi) = v^{\lambda,\alpha}(N^*).$$

Thus, these vectors satisfy (3), (4) and (5) for $v^{\lambda,\alpha}$. By induction hypothesis

$$\lambda_i b_i[\emptyset] = Sh_i\left(N^*, v^{\lambda,\alpha}\right) = Sh_i\left(N \setminus \{\alpha\}, v^{\lambda,\alpha}\right).$$

We now have

$$\begin{aligned} n \lambda_i a_i(\emptyset) &= \sum_{j \in N} \lambda_i a_i([j]) = \lambda_i a_i([i]) + \sum_{j \in N \setminus \{i\}} \lambda_i a_i([j]) \\ &= \lambda_i a_i([i]) + \sum_{j \in N \setminus \{i\}} Sh_i\left(N \setminus \{j\}, v^{\lambda,j}\right). \end{aligned}$$

Let (N^*, w^*) be the inessential form given by $w^*(T) = \sum_{i \in T} \lambda_i a_i(\emptyset)$ for all $T \subset N^*$. Then, $v^{\lambda,j} = \rho w^* + (1-\rho) v^\lambda$. By the additivity property of the Shapley value:

$$\begin{aligned} Sh_i\left(N\backslash\{j\}, v^{\lambda,j}\right) &= \rho Sh_i\left(N\backslash\{j\}, w^*\right) + (1-\rho) Sh_i\left(N\backslash\{j\}, v^\lambda\right) \\ &= \rho\lambda_i a_i(\emptyset) + (1-\rho) Sh_i\left(N\backslash\{j\}, v^\lambda\right) \end{aligned}$$

and thus

$$\begin{aligned} n\lambda_i a_i(\emptyset) &= \lambda_i a_i([i]) + (n-1)\rho\lambda_i a_i(\emptyset) + (1-\rho) \sum_{j \in N\backslash\{i\}} Sh_i\left(N\backslash\{j\}, v^\lambda\right) \\ &= \lambda_i\left[a_i([i]) - \rho a_i(\emptyset)\right] + n\rho\lambda_i a_i(\emptyset) + (1-\rho) \sum_{j \in N\backslash\{i\}} Sh_i\left(N\backslash\{j\}, v^\lambda\right). \end{aligned}$$

By Lemma 5.3,

$$n(1-\rho)\lambda_i a_i(\emptyset) = (1-\rho)\left[v^\lambda(N) - v^\lambda(N\backslash\{i\})\right] + (1-\rho) \sum_{j \in N\backslash\{i\}} Sh_i\left(N\backslash\{j\}, v^\lambda\right)$$

and thus

$$n\lambda_i a_i(\emptyset) = v^\lambda(N) - v^\lambda(N\backslash\{i\}) + \sum_{j \in N\backslash\{i\}} Sh_i\left(N\backslash\{j\}, v^\lambda\right) = nSh_i\left(N, v^\lambda\right).$$

■

6 The main conjectures

We have the following conjecture (cf. Proposition 6 in Hart and Mas-Colell (1996)):

Conjecture 6.1 *Suppose that (N, V) is a form satisfying (A1)-(A5). Then, for each $0 \leq \rho < 1$ there exists an equilibrium.*

We also have (cf. Proposition 7 in Hart and Mas-Colell (1996)):

Conjecture 6.2 *Suppose that* (N, V) *is a form satisfying (A1)-(A5) and, moreover,* $\partial V(N)$ *is flat. Then, for each* $0 \leq \rho < 1$ *there is a unique equilibrium. Moreover, the expected equilibrium payoff equals the unique Shapley NTU value payoff of* (N, V).

Notice that the game for $\rho = 0$ is (essentially) the game presented by Vidal-Puga (2004), that yields the Shapley NTU value when $V(N)$ is flat. Let $\lambda \in \mathbb{R}^N_{++}$ be the outward orthogonal vector to $V(N)$. In the game $A(N, \emptyset)$, a random permutation π is chosen, and the subgames $B(S, \pi, f)$ with $S = N \backslash \{\pi_1, ..., \pi_t\}$ are the stages in Vidal-Puga (2004). Given an order π, the final equilibrium payoff is $\frac{1}{\lambda} m(\pi, v^\lambda)$.

Finally, we have (cf. Proposition 8 in Hart and Mas-Colell (1996)):

Conjecture 6.3 *Let* (N, V) *be a form satisfying (A1)-(A4). If* $a(\rho)$ *is an equilibrium payoff for each* ρ *and* a *is a limit point of* $a(\rho)$ *as* $\rho \to 1$, *then* a *is a Shapley NTU value payoff of* (N, V).

The forms with $\partial V(N)$ flat in our model take the role of the hyperplane forms in Hart and Mas-Colell's model.

7 Two examples

7.1 A classical example

In this section we apply the above game to an exchange economy which appeared in Hart (1985):

Example 7.1 *(Hart, 1985) Consider a pure exchange economy with three traders and three commodities. Initial endowments and utility functions are given by:*

trader	*endowment*	*utility function*
1	$(2, 2, 0)$	$a + b - 4$
2	$(2, 0, 2)$	$\frac{a}{2} + c - 3$
3	$(0, 2, 2)$	$b + c - 4$

The NTU form is given by

$$\begin{aligned}
V(\{i\}) &= \left\{t \in \mathbb{R}^{\{i\}} : t \leq 0\right\} \text{ for } i \in N \\
V(\{1,2\}) &= \left\{(t_1,t_2) \in \mathbb{R}^{\{1,2\}} : t_1 + 2t_2 \leq 0, 2t_1 + t_2 \leq 3\right\} \\
V(\{i,3\}) &= \left\{(t_i,t_3) \in \mathbb{R}^{\{1,3\}} : t_i + t_3 \leq 0\right\} \text{ for } i = 1,2 \\
V(N) &= \left\{(t_1,t_2,t_3) \in \mathbb{R}^{N} : t_1 + t_2 + t_3 \leq 1\right\}.
\end{aligned}$$

The Shapley NTU value proposes a payoff of $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

The TU game associated with this example is given by $\lambda = (1,1,1)$ and $v^{\lambda}(\{i\}) = 0$, $v^{\lambda}(\{1,2\}) = 1$, $v^{\lambda}(\{i,3\}) = 0$, and $v(N) = 1$ (see Figure 2.)

Let $a \in V(N)$ be an equilibrium payoff.

Assume the first player to be chosen is player 3. Players 1 and 2 should decide an offer f. If player 3 disagrees on f he would counteroffer g^f with:

$$\begin{aligned}
g_1^f(N) &= \rho a_1 + (1-\rho) f_1(\{2,3\}) \\
g_2^f(N) &= \rho a_2 + (1-\rho) f_2(\{2,3\}) \\
g_3^f(N) &\in \partial V(N)
\end{aligned}$$

Hence, we have

$$g_3^f(N) = 1 - \rho(a_1 + a_2) - (1-\rho)(f_1(\{2,3\}) + f_2(\{2,3\})).$$

The final payoff for player 3 is $g_3^f(N)$. Then, players 1 and 2 should propose f such that $f_3(N) = g_3^f(N)$, i.e.

$$f_3(N) = 1 - \rho(a_1 + a_2) - (1-\rho)(f_1(\{2,3\}) + f_2(\{2,3\})).$$

Then, player 3 is bound to agree and the final payoff is given by $f(N)$. The optimal agreements for players 1 and 2 are those which give player 3 the minimum possible, i.e.

$$\begin{aligned}
f_3(N) &= \min_{(t_1,t_2) \in V(\{1,2\})} \{1 - \rho(a_1 + a_2) - (1-\rho)(f_1(\{2,3\}) + f_2(\{2,3\}))\} \\
&= 1 - \rho(a_1 + a_2) - (1-\rho) \max_{(t_1,t_2) \in V(\{1,2\})} \{(f_1(\{2,3\}) + f_2(\{2,3\}))\} \\
&= 1 - \rho(a_1 + a_2) - (1-\rho) \\
&= \rho(1 - a_1 - a_2)
\end{aligned}$$

which is attainable by proposing f with $f(\{1,2\}) = (2,-1)$.

If players 1 and 2 do not reach an agreement, their final payoff will be $(\rho a_1, \rho a_2)$. Then, when deciding an offer, players 1 and 2 are essentially playing a Rubinstein alternating-offer model in a pure bargaining problem (D,d) with disagreement point $d = (\rho a_1, \rho a_2)$ and

$$D = \{(t_1,t_2) : t_1 + t_2 \leq 1 - \rho(1 - a_1 - a_2)\}.$$

It is well-know that the expected final payoff (when player 1 is chosen first) will be given by

$$\begin{aligned} b_1 &= \frac{1 + \rho a_1 - \rho a_2 - \rho(1 - a_1 - a_2)}{2} \\ b_2 &= \frac{1 - \rho a_1 + \rho a_2 - \rho(1 - a_1 - a_2)}{2} \\ b_3 &= \rho(1 - a_1 - a_2) \end{aligned}$$

Assume now the first player in being chosen is player 2. Then, this player 2 would assure a payoff of $1 - \rho(a_1 + a_3)$ by disagreeing. The final payoff will be given then by $1 - \rho(a_1 + a_3)$ for player 2 and ρa_j for the other two players.

Analogously, when the first player in being chosen is player 1, the final payoff is $1 - \rho(a_2 + a_3)$ for player 1 and ρa_j for the other two players.

Then, the final payoff is given by

$$\begin{aligned} a_1 &= \frac{(b_1) + (1 - \rho(a_2 + a_3)) + (\rho a_1)}{3} \\ a_2 &= \frac{(b_2) + (1 - \rho(a_1 + a_3)) + (\rho a_2)}{3} \\ a_3 &= \frac{(b_3) + (\rho a_3) + (\rho a_3)}{3}. \end{aligned}$$

whose unique solution (for any ρ) is $(0.5, 0.5, 0)$.

7.2 An example with general $\partial V(N)$

We consider now an example in which $\partial V(N)$ is not flat. Let $N = \{1,2,3\}$ and let V be given by

$$\begin{aligned} V(\{i\}) &= \{(t_i) : t_i \leq 0\} \text{ for } i = 1,2,3 \\ V(\{1,2\}) &= \{(t_1,t_2) : t_1 + t_2 \leq 1, t_1 \leq 1, t_2 \leq 1\} \\ V(\{i,3\}) &= \{(t_i,t_3) : t_i \leq 0, t_3 \leq 0\} \text{ for } i = 1,2 \\ V(N) &= \left\{(t_1,t_2,t_3) : t_1^2 + t_2^2 + t_3^2 \leq 1\right\} - \mathbb{R}_+^N. \end{aligned}$$

This NTU form is represented in Figure 3. The only Shapley NTU value is $(2/3, 2/3, 1/3)$ for $\lambda = (2,2,1)$.

Let $a \in V(N)$ be an equilibrium payoff.

Assume the first player to be chosen is player 3. Players 1 and 2 should decide an offer f with $f(\{1,2\}) = (u,v)$ such that z with

$$(\rho a_1 + (1-\rho)u)^2 + (\rho a_2 + (1-\rho)v)^2 + z^2 = 1$$

is minimum. This minimum is reached with $(u,v) = (1,0)$ or $(u,v) = (0,1)$. For $(u,v) = (1,0)$:

$$z = \sqrt{1 - (\rho a_1 + 1 - \rho)^2 - \rho^2 a_2^2}$$

for $(u,v) = (0,1)$:

$$z = \sqrt{1 - \rho^2 a_1^2 - (\rho a_2 + 1 - \rho)^2}. \tag{9}$$

We assume $a_1 \leq a_2$. Then, the minimum is (9).

Now, players $1,2$ play a pure bargaining problem (D,d) with $d = (\rho a_1, \rho a_2)$ and

$$D = \left\{(x,y) : x^2 + y^2 \leq 1 - z^2\right\}.$$

Then, player 1 proposes (x,y) such that

$$\begin{aligned} x &= \rho b_1 + (1-\rho)\rho a_1 \\ y &= \sqrt{1 - z^2 - (\rho b_1 + (1-\rho)\rho a_1)^2} \end{aligned}$$

and player 2 proposes (x, y) such that

$$\begin{aligned} x &= \sqrt{1 - z^2 - (\rho b_2 + (1-\rho)\,\rho a_2)^2} \\ y &= \rho b_2 + (1-\rho)\,\rho a_2. \end{aligned}$$

Then, the expected final payoff (when player 3 is chosen first) is

$$b_1 = \frac{\rho b_1 + (1-\rho)\,\rho a_1 + \sqrt{1 - z^2 - (\rho b_2 + (1-\rho)\,\rho a_2)^2}}{2} \tag{10}$$

$$b_2 = \frac{\rho b_2 + (1-\rho)\,\rho a_2 + \sqrt{1 - z^2 - (\rho b_1 + (1-\rho)\,\rho a_1)^2}}{2} \tag{11}$$

$$b_3 = z. \tag{12}$$

When player 2 is chosen first, the final payoff is $(\rho a_1, \rho a_3)$ for $\{1, 3\}$, and player 2 gets y such that $(\rho a_1, y, \rho a_3) \in \partial V(N)$. Hence the final payoff (when player 2 is chosen first) is

$$\left(\rho a_1, \sqrt{1 - \rho^2\,(a_1^2 + a_3^2)}, \rho a_3\right).$$

Analogously, when player 1 is chosen first, the final payoff is

$$\left(\sqrt{1 - \rho^2\,(a_2^2 + a_3^2)}, \rho a_2, \rho a_3\right).$$

Hence, the final expected payoff is

$$a_1 = \frac{b_1 + \sqrt{1 - \rho^2\,(a_2^2 + a_3^2)} + \rho a_1}{3} \tag{13}$$

$$a_2 = \frac{b_2 + \sqrt{1 - \rho^2\,(a_1^2 + a_3^2)} + \rho a_2}{3} \tag{14}$$

$$a_3 = \frac{b_3 + 2\rho a_3}{3}. \tag{15}$$

The only solution of (9)-(15) is given[8] in the following table:

[8] Solved using Scientific WorkPlace 5.00 (MacKickan Software, Inc.)

ρ	a_1	a_2	a_3	$\sqrt{a_1^2+a_2^2+a_3^2}$
0	0.5	0.5	0	0.707 11
0.5	0.614 84	0.614 84	0.251 78	0.905 23
0.9	0.664 08	0.664 08	0.329 18	0.995 17
0.99	0.666 64	0.666 64	0.333 29	0.999 95

We compare this convergence to the convergence of the mechanism of Hart and Mas-Colell. The consistent value is also $(2/3, 2/3, 1/3)$:

ρ	a_1	a_2	a_3	$\sqrt{a_1^2+a_2^2+a_3^2}$
0	0.5	0.5	0.235 7	0.745 36
0.5	0.595 72	0.595 72	0.316 11	0.899 83
0.9	0.660 44	0.660 44	0.343 10	0.995 03
0.99	0.666 28	0.666 28	0.334 72	0.999 95

8 A brief discussion

In both the Shapley NTU value and the consistent value, the payoff that the subcoalitions would get by themselves play an important role. However, in the consistent value, these payoffs are computed assuming that the other players are not present. Notice that this feature is presented in the game of Hart and Mas-Colell's (1996), where subcoalitions reach agreements when the rest of the players are not present (have been dropped out). See Figure 4a). These potential agreements are used as a threat in the negotiations among the agents. The transfer rates are different depending on the subcoalition.

On the other hand, the transfer rate of each subcoalition in the Shapley NTU value is computed taking into account the grand coalition, i.e. even though players inside subcoalitions bargain among them, they do so taking into account that the other players are still present. This feature also occurs in our game, where the final agreement is reached with all players present. This difference between both models should be stressed, because our game do not exclude any player.

One can imagine a group of people discussing about how to share the benefit of their cooperation by doing offers and counteroffers. Even though these offers may be given by single individuals (as in Hart and Mas-Colell's) one can imagine that the offers come from coalitions.

Hart and Mas-Colell's model rules out this possibility. The role of the coalitions only arises when the rest of the players are dropped out. Excluded players have no chance to come back to the negotiation table. This fact is quite problematic. In particular, since there is no planner, what does impede the excluded players to negotiate among them?

Alternatively, one may think that coalitions should be capable to agree on a common offer. If the coalition is a singleton, it is clear that its single member would make an optimal offer for his interests. For two-player coalitions, it is not so clear how they can agree on an optimal offer, because they have to balance their mutual interests. However, the economic theory provides us with a tool for solving the two-player case. The tool is the alternating offers model of Rubinstein[9] (1982). For three players, the offers are done by single players or by two-player coalitions, and so on. Once we have a game for coalitions of less than n players, we can apply an alternating-offers model to bargain in the n-player case.

Our model follows this philosophy.

9 The limit equilibrium payoff

In this Section, we study property (A5). This property is only used in Conjecture 6.1. For example, consider the form (N, V) with $n = 3$ given by

$$\begin{aligned} V(\{i\}) &= \{(x_i) : x_i \leq 0\} \text{ for all } i \in N \\ V(\{1,2\}) &= \left\{(x_1, x_2) : x_2 + \frac{x_1^3 + x_1 + 1}{x_1^2 + 1} \leq 3, x_2 \leq \frac{\sqrt{3}}{3}\right\} \\ V(\{1,i\}) &= \{(x_1, x_i) : x_1, x_i \leq 0\} \text{ for } i = 2, 3 \\ V(N) &= \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 4\}. \end{aligned}$$

[9]Other possible models are provided by Luce and Raiffa (1953) and Moulin (1984).

This form is sketched in Figure 1.

There is no equilibrium for this form, because players 1 and 2 have no best strategy when deciding an offer. However, we can define the limit equilibrium as follows.

Given $\varepsilon > 0$, a strategy profile is an ε-*equilibrium* if the agents play an ε-optimal strategy in each subgame, i.e. no player can improve his payoff more than ε by deviating in any subgame. Notice that any equilibrium is an ε-equilibrium.

Assume for each $\varepsilon > 0$ we have an ε-equilibrium and the ε-equilibrium payoffs approach a as ε approaches 0. Then, we say that a is a *limit equilibrium payoff.* Of course, any equilibrium payoff is a limit equilibrium payoff.

Now, we can restate Conjecture 6.1 as follows:

Conjecture 9.1 *Suppose that (N, V) is a form satisfying (A1)-(A4). Then, for each $0 \leq \rho < 1$ there exists a limit equilibrium payoff.*

And we can restate Conjecture 6.2 as follows:

Conjecture 9.2 *Suppose that (N, V) is a form satisfying (A1)-(A4) and, moreover, $V(N)$ is delimited by a hyperplane. Then, for each $0 \leq \rho < 1$ there is a unique limit equilibrium payoff. Moreover, the limit equilibrium payoff equals the unique Shapley NTU value payoff of (N, V).*

10 Union structures

Frecuently, players do not act independently, but they belong to an union of agents who act together, as one unit, relative to the rest of the players. Take for example political parties in a parliament, syndicates in a wave negotiation, parliamentary coalitions in a government, firms in a cartel, countries in a tariff bargaining, member states of a federated country, etc. While the individual players still play their roles as decision makers inside a union, in all interactions with the other players, union members act as one unit.

Let (N, V) be a form. A *union structure*[10] is a partition $P = \{S_1, S_2, ..., S_p\}$ of N, i.e. $\bigcup_{q=1}^{p} S_q = N$ and $S_q \cap S_r = \emptyset$ for all $q \neq r$. Then, (N, V, P) is a form with union structure.

Assume (N, v, P) is a TU form with union structure. The *form between unions* $(v, v/P)$ is defined as follows:

$$(v/P)(B) = \sum_{S_q \in B} v(S_q)$$

for all $B \subset P$.

The *Owen value* (Owen, 1977) of a TU form v on N with union structure P is the vector $Ow(N, v, P) \in \mathbb{R}^N$ whose ith coordinate is given by

$$Ow_i(N, v, P) \equiv \frac{1}{|\Pi(P)|} \sum_{\pi \in \Pi(P)} m_i(\pi, v) \in \mathbb{R}$$

where $\Pi(P) \subset \Pi$ is the set of permutations in which the unions in P come together.

In TU forms with union structure, the Owen value is a relevant solution concept. It has been supported axiomatically (Owen (1977), Hart and Hurz (1983, 1984), Winter (1992), Calvo et al. (1996), among others) and also non-cooperatively (Vidal-Puga and Bergantiños (2003), Vidal-Puga (2005)). Moreover, it has been successfully applied to cost allocation problems (Vázquez-Brage et al. (1997)) and political situations (Carreras and Owen (1988, 1993), Ono and Muto (2001)).

In NTU forms with union structures, several generalizations of the Owen value have been proposed. Winter (1991) presents the *game coalition structure solution*, which generalize both the Owen value and the Harsanyi value (Harsanyi (1963)). Bergantiños and Vidal-Puga (2004) present the *consistent coalitional value* and the *random order value*, which generalize both the Owen value and the consistent value (Maschler and Owen (1989, 1992)).

[10] In the literature, the most frequent name is *coalition structure*. We use the term *union* in order to avoid ambiguities with a coalition of players.

Krasa, Temimi and Yannelis (2003) first study the Owen NTU value[11], which generalize both the Owen value and the Shapley NTU value.

A point $x \in V(N)$ is a *Owen NTU value* payoff of V if there exists a vector $\lambda \in \mathbb{R}^N_{++}$ such that $\lambda_i x_i = Ow_i\left(N, v^\lambda\right)$ for all $i \in N$.

We can easily modify our game so that an union structure is taken into account.

We call a *supercoalition* any nonempty subset of P. Hence, for a TU form (N, v, P) with union structure, a supercoalition is a coalition in the TU form between unions $(P, v/P)$.

Assume supercoalition $B \subset P$ should decide an offer. If $B = \{S_q\}$, then union S_q proposes a rule following the game given in Section 3.

Suppose we know how a supercoalition of size $|B| - 1$ decides an offer. We now define how supercoalition B does. First, a union S_q is randomly chosen out of B, being each union equally likely to be chosen. The supercoalition $B \backslash \{S_q\}$ decides an offer f (we know how, by induction hypothesis). Players in S_q can either agree or disagree on f. They do so by voting sequentially in a random order[12]. If all of them agree, then f will be the offer of supercoalition B. In case one of them disagrees, then the union S_q counterproposes a rule g, following the game given in Section 3. If all the unions in $B \backslash \{S_q\}$ accept (their members are asked in some random order[13]) then g will be the offer of supercoalition B. If at least a coalition in $B \backslash \{S_q\}$ rejects, then with probability ρ a new union is randomly chosen out of B, and the process repeats. With probability $1 - \rho$, there is breakdown and the offer of supercoalition B will be given by $h(T) = f(T)$ if $T \cap S_q = \emptyset$ and $h(T) = \left(f\left(T \backslash (T \cap S_q)\right), \omega_{T \cap S_q}\right)$ if $T \cap S_q \neq \emptyset$.

[11]Shapley (1969) presented the Shapley NTU value as a generalization of any value in TU games to NTU games. However, his generalization has been mainly used for the Shapley value. Krasa et al. work with this generalization for the Owen value.

[12]One can define this voting stage as in the mechanism of Section 3. Instead of rules, players propose answers, 'to agree' or 'to disagree'. In case of breakdown, the answer is 'to disagree'. We use the sequential vote protocol because it is simpler to analyze.

[13]The same as in the previous footnote.

In the case the supercoalition is P, then the proposed rule f is implemented, i.e. each player i receives $f_i(N)$.

Then, we have the same conjectures as in Section 6 for the Owen NTU value:

Conjecture 10.1 *Suppose that* (N, V, P) *is a form with union structure such that* (N, V) *satisfies properties (A1)-(A5). Then, for each* $0 \leq \rho < 1$ *there is an equilibrium. Moreover, as* ρ *approaches* 1*, every limit point of equilibrium payoff is an Owen NTU value of* (N, V, P)*.*

The idea of the proof is as follows. Assume $\rho = 0$ and $\partial V(N)$ is flat. Assume the unions are randomly chosen in the order $S_p, S_{p-1}, ..., S_2$ and thus union S_1 should make the offer. Assume $S_1 = \{1, 2, ..., s_1\}$. Assume the players in S_1 are randomly chosen in the order $s_1, s_1 - 1, ..., 1$ and thus player 1 should make the offer. Then, the final payoff is $\frac{1}{\lambda} \sum_{\pi \in \Pi'} m\left(\pi, v^{\lambda}\right)$ where Π' is the set of permutations compatible with the order $[1, 2, ..., s_1, S_2, ..., S_p]$. The following feature is important: the player who potentially disagrees is not the one that makes the counteroffer. The order in which this counteroffer is decided is chosen randomly among the permutations of the members of S_r, so by expected terms the members of S_r get what player 1 is offering to them.

Hence, the final expected payoff is given by the average of the vectors of marginal contributions when the players of the same coalition come together, i.e. the Owen value of v^{λ}.

Remark 10.1 *Vidal-Puga (2005) generalizes Hart and Mas-Colell's model so that a union structure is taken into account. The generalized game works well for TU forms and pure bargaining problems. For general NTU forms, the resulting equilibrium payoff, if efficient, is the consistent coalitional value (Bergantiños and Vidal-Puga (2004)). However, the result is not completely satisfactory, since in many NTU forms the equilibrium payoff is inefficient.*

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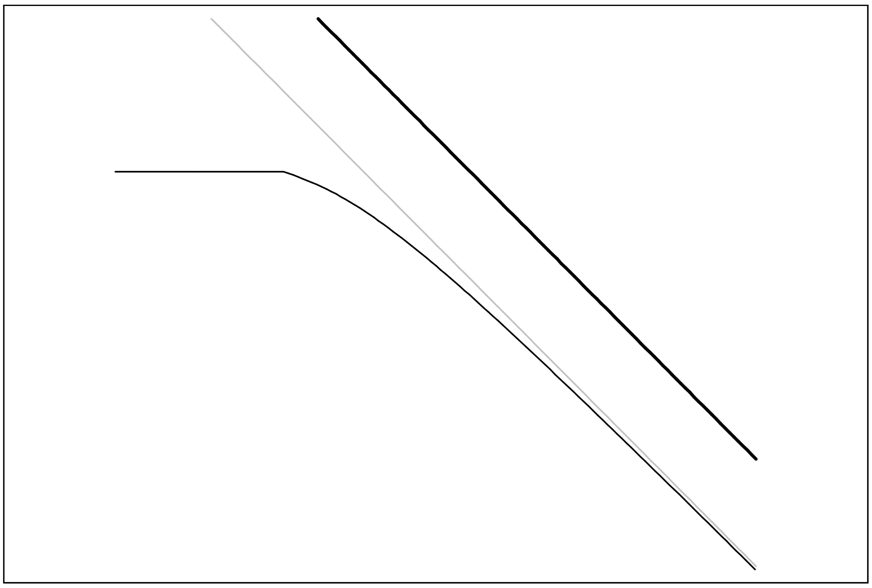

Figure 1: The thick line represents the projection of $\partial V(N)$. The thin line represents $\partial V(S)$.

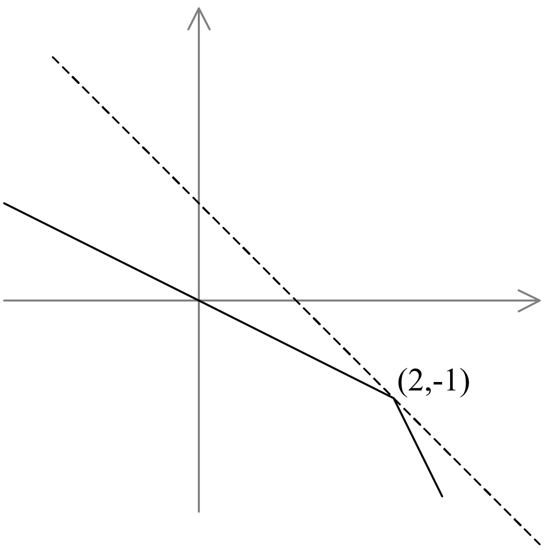

Figure 2: The set $V(\{1,2\})$.

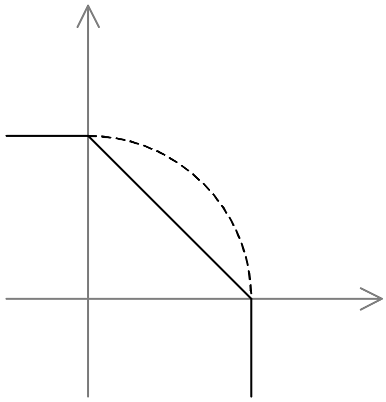

Figure 3: The set $V(\{1,2\})$.

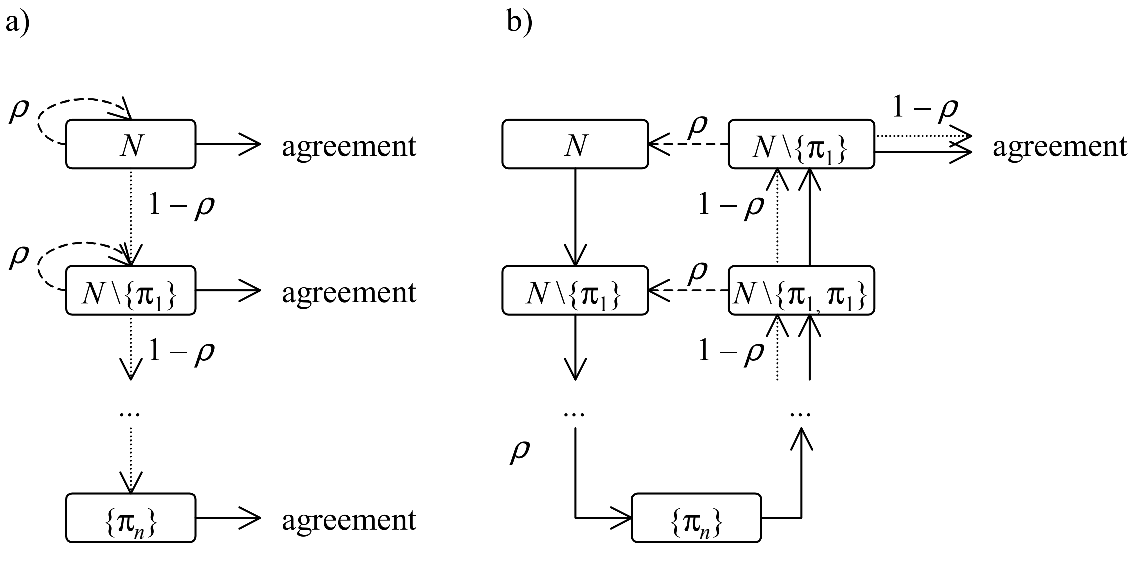

Figure 4: a) Hart and Mas-Colell's (1996) model; b) our model.