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Nash equilibria match up with Radner equilibria

Guadalupe Fugarolas, Carlos Hervés-Beloso, Emma Moreno-García y Juan Pablo Torres-Martínez

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Facultade de Ciencias Económicas e Empresariais, Campus As Lagoas-Marcosende, 36310 Vigo.

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Guadalupe Fugarolas

Facultad de Económicas. Universidad de Vigo.

e-mail: gfa@uvigo.es

Carlos Hervés-Beloso

RGEA. Facultad de Económicas. Universidad de Vigo.

e-mail: cherves@uvigo.es

Emma Moreno-García

Facultad de Economía y Empresa. Universidad de Salamanca.

e-mail: emmam@usal.es

Juan Pablo Torres-Martínez

Pontifícia Universidade Católica do Rio de Janeiro

e-mail: jptorres-martinez@econ.puc-rio.br

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Abstract. In this paper we recast a differential information economy as a game in normal form. We show that in this game Nash equilibria are strong and coincide with the Walrasian expectations equilibria of the underlying economy.

Key words: Differential information economy, strategic market games, Nash equilibrium, strategic outcome functions, competitive private equilibrium, non-free disposal.

1 Introduction.

The first studies of games in the economics literature were the papers by Cournot (1838), Bertrand (1883) and Edgeworth (1925) but these were seen as special models that did little to change the way economists thought about most problems. The idea of a general theory of games was introduced by von Neumann and Morgenstern in their famous 1944 book "Theory of Games and Economic Behaviour", which proposed that most economic questions should be analized as games. Nash (1950) established what came to be known as the noncooperative concept of *Nash equilibrium*. This is a natural generalization of the equilibria studied in specific models by Cournot and Bertrand where the strategies of the players are simply their choices of quantities and prices respectively, and it is the starting point for most economic analyses.

Since Nash contribution to the existence of equilibrium points in noncooperative games there has been a growing literature on the strategic approaches to economic equilibrium. Arrow and Debreu (1954) and Debreu (1962) extended Nash's model to generalized games: by adding a fictitious price player whose payoff function is the value of excess demand, they considered an abstract economy and a social equilibrium. Walras equilibria were obtained then as Nash equilibria of a pseudo-game that included the market participant.

The formation of prices plays a central role in any discussion of the market process: which of the given economic agents sets the price vector and in what way? This question gives rise to a series of papers on strategic market games where, in addition to the consumer behavior, a price mechanism is formulated. Several types of price-forming mechanisms have been described in which prices depend on the actions of the traders, avoiding the classical assumption that individuals must regard prices as fixed.

The majority of these works proposed strategic market game models of exchange economies in the spirit of Cournot and Bertrand to study the relationship between the Nash equilibria and the competitive equilibria. On one hand, and following the Cournot tradition, Shapley (1976), Shapley and Shubik (1977), Dubey and Shapley (1994) and Dubey and Geanakoplos (2003) proposed games where money is introduced as the stipulated medium of exchange, either as one of the intrinsically valuable commodities or as "inside" fiat money. These games adhere completely to the Nash format, i.e., no price player is involved, and it is shown that their Nash equilibria converge to Walras equilibria.

On the other hand, the analysis of Bertrand was extended by Schmeidler (1980) and Dubey (1982) who established the coincidence of Nash equilibria and Walras equilibria. In particular, Schmeidler (1980) provided a strategic market game model where the exchange mechanism is characterized by a strategy outcome function that maps agents selections of strategies to allocations. For it, it was proposed a game in strategic form in which the choice of a price vector is a part of the strategy choice for each player and proved that the Nash equilibrium of the game is precisely the competitive equilibrium of the Arrow-Debreu pure exchange economy.

The purpose of this paper is to extend this Schmeidler result to economies in which differences in information among economic agents are allowed and, in particular, to analyze the strategic approach to Walrasian expectations equilibrium.

We begin with the model of a differential information economy. Radner (1968) was the first in introducing differential information into a model of general equilibrium. To cover the case of private information, Radner extended Debreu's analysis of uncertainty. Debreu (1959) had already introduced uncertainty in his model by considering a finite set of states of nature and viewing the commodities as differentiated by state. Radner then allows the agents to be asymmetrically informed about the states of nature: every agent is assigned a partition of the set of states of nature, with the interpretation that the sets in the partition contain those states which cannot be distinguished by the agent. In a Radner-type economy with differential information it is assumed that if an agent is incompletely informed about the states of nature he can only carry out such consumptions that are compatible with her information, that is, she cannot act differently on states he cannot distinguish.

Radner's model is an extension of the original model of Arrow-Debreu: in the special case where there is no uncertainty or if all agents have the same information at their disposal, then the standard Walras exchange economy will result. Almost all studies of differential information economies conform to the Radner approach and assume that the agent's preferences can be described by an expected utility function. However, as Debreu explains in Chapter 7 of his "Theory of Value" this representation of preferences is not essential. Instead, one can consider preferences defined for state-dependent commodities and the preference relation itself reflects the personal estimation of the probability that a certain state occurs. Recently, Maus (2003, 2004) proposed a differential information economy model where preferences are described by a utility function which maps the state-commodity space to the reals. This model is able to incorporate the expected utility case and is the one which we will consider. The starting point of our work is this more general model of differential information economy without expected utilities. We firstly define a suitable economy where private information in the original economy is introduced in each agent's consumption set. For this new associated economy, we introduce the notion of non-free disposal competitive equilibrium in the sense of Radner, here called Walrasian expectations equilibrium or competitive private equilibrium.

The notion of Walrasian equilibrium in these economies was introduced by Radner (1968). This noncooperative solution concept presumes that decisions are made in a ex-ante stage and agents maximize their utility subject to their budget constraints, where information contraints, besides the classical ones are considered. It should be stressed that each agent's competitive allocation is measurable with respect to his own private information so all choices made reflect the informational asymmetries.

Precisely, in our extension of Schmeidler analysis to the case of differential information, the main difficulty to overcome is that the outcomes an agent receives have to be measurable with respect to her information partition. In fact, even if in the strategy set of an agent is considered a vector that it is compatible with her information, the strategic outcome function could lead to outcome bundles that are not measurable. This suggests us to consider a variant of the Schmeidler game whose strategic outcome functions give each agent a consumption plan that is compatible with his private information. In addition, it is shown that Nash equilibria of our game are strong and coincide with the competitive private equilibria of the underlying private information economy.

Radner equilibrium has been extensively researched as regards existence: it was proved that, under usual assumptions which guarantee the existence of Walras equilibria, Radner equilibria exist [see Radner (1968,1982) and Maus (2004) for finite economies with exchange and production and Einy-Moreno-Shitovitz (2001) for exchange economies with a continuum of traders]. All these existence results but Maus (2004) rely on the free disposal, i.e., for some state of nature the total consumption does not exceed the total endowment, to guarantee the equilibrium existence with non-negative prices. As we will see in Section 2 agents' lack of information on certain states implies some form of free disposal. In Theorem 2.1 we show that if every state of nature can be identified by at least one agent in the economy, then any Radner equilibrium is with non-free disposal and prices are positive. The non-free disposal competitive private equilibrium seems to be an appealing concept and is the one we deal with in this article.

The rest of the paper is organized as follows. In Section 2 we set the basic formal model of a differential information model. We present a description of a suitable economy associated with it. The central non cooperative competitive private equilibrium notion used in this paper is introduced and we discuss its existence. In Section 3 we remodel the associated economy as a game in normal form where not only prices but also quantities are set by agents; the main result and its proof are presented.

2 The model: a private information economy.

We consider an exchange economy E with differential information. The economy consists of a finite set of agents and states of nature and there is a finite number of goods, l, in each state.

Let $\Omega = \{\omega_1, \dots, \omega_k\}$ be the finite set of states of the world. Let $\mathcal{P}$ be the set of partitions of Ω. An information set is a partition of Ω, that is, a $P \in \mathcal{P}$. The interpretation is that the states contained in an element $S \in P$ cannot be distinguished when the information set is P. For each $\omega \in \Omega$, denote by $P(\omega)$ the element of the partition that contains ω, i.e., $P(\omega)$ contains those states that cannot be distinguished from ω under the information P.

The consumption set for every agent is $(\mathbb{R}^\ell_+)^k$ which denotes the positive orthant of $(\mathbb{R}^\ell)^k$. A consumption bundle is a vector $a = (a(\omega_1), \dots, a(\omega_k))$ $= (a(\omega))_{\omega\in\Omega} \in (\mathbb{R}^\ell_+)^k$, where $a(\omega) \in \mathbb{R}^\ell_+$ denotes the consumption of the ℓ goods in each state $\omega \in \Omega$.

An information set P generates a σ-algebra $\sigma(P)$. A commodity bundle $a : \Omega \to \mathbb{R}^\ell_+$ is P-measurable when is $\sigma(P)$-measurable, that is, when it is constant on the elements of P so $a(\omega) = a(\omega')$ for all $\omega' \in P(\omega)$.

Let $N = \{1, \dots, n\}$ be the set of agents. Each agent $i \in N$ is characterized by her private information set $P_i \in \mathcal{P}$, her initial endowments $e_i \in (\mathbb{R}^\ell_+)^k$, that are supposed to be P_i-measurable, and her preference relation over $(\mathbb{R}^\ell_+)^k \times (\mathbb{R}^\ell_+)^k$ represented by a utility function $U_i : (\mathbb{R}^\ell_+)^k \to \mathbb{R}$.

Therefore, a differential information exchange economy E is defined by

$$E = \left((\mathbb{R}^\ell_+)^k, (U_i, e_i, P_i), i = 1, ..., n\right).$$

An allocation x in an economy E is a commodity bundle $x_i \in (\mathbb{R}_+^\ell)^k$ for every i. An allocation $x : N \times \Omega \to \mathbb{R}_+^\ell$ is physically feasible in E if $\sum_{i=1}^{n} x_i(\omega) \leq \sum_{i=1}^{n} e_i(\omega)$, for each $\omega \in \Omega$. An allocation $x = (x_1, \ldots, x_n)$ is informationally feasible if x_i is P_i-measurable for every i. An allocation is said to be feasible in an economy E if it is physically and informationally feasible.

Note that the consumption sets for every agent in this differential information economy E coincide with the positive orthant of the commodity space but private information of each agent places a restriction on the consumption.

Let $\Delta = \left\{ p \in (\mathbb{R}_+^\ell)^k \mid \sum_{h=1}^{\ell \times k} p_h = 1 \right\}$. A price system is an element $p \in \Delta$ that specifies a price $p(\omega) \in \mathbb{R}_+^l$ for the l goods in each state $\omega \in \Omega$. For a price system $p : \Omega \to \mathbb{R}_+^\ell$, the budget set of agent i is given by

$$\begin{aligned} B_i(p) \;\; = \;\; & \{x_i \in (\mathbb{R}_+^\ell)^k \mid x_i \text{ is } P_i - \text{measurable and} \\ & p \cdot x_i = \sum_{\omega \in \Omega} p(\omega) \cdot x_i(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega) = p \cdot e_i\}. \end{aligned}$$

We firstly note that the budget set is across the states of nature, i.e., in average. Secondly, the budget set includes informational constraint: better information allows for more contingent trades and enlarges her budget set.

Definition 2.1 *A Walrasian expectations equilibrium for an economy E is a pair (x, p), where $x = (x_1, ..., x_n)$ is a feasible allocation and p is a price system, such that x_i maximizes U_i on $B_i(p)$ for all $i \in N$.*

As mentioned in the introduction the model considered here is able to incorporate models which use an expected utility function. Radner's (1982) definition of equilibrium in a differential information economy sets the additional condition

$$\sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^{n} x_i(\omega) = \sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^{n} e_i(\omega)$$

which is also considered in more recent works (see Glycopantis, Muir y Yannelis (2003) and Glycopantis y Yannelis (2004)). This condition ensures that if a commodity is in excess supply its price is zero. Actually, this condition is a particular case of Walras' Law.

As it is well known, Walras' Law states that for any price system and not only for equilibrium prices, the aggregate demand value is equal to the total initial endowment value. In fact, if preferences are weakly monotone ($x \ll y \ \Rightarrow\ U_i(x) < U_i(y)$) the economy satisfies Walras' Law. For it, let $p \in \Delta$ be a price system and let $x_i(\omega)$ denote the i^{th} agent demand at state ω and prices p. Then the budget set of agent i is saturated, i.e., $p \cdot x_i = p \cdot e_i$ because otherwise there would exist an agent j such that

$$p \cdot x_j = \sum_{\omega\in\Omega} p(\omega) \cdot x_j(\omega) < \sum_{\omega\in\Omega} p(\omega) \cdot e_j(\omega) = p \cdot e_j$$

and there will be a neighborhood V of x_j such that $p \cdot z < p \cdot e_j$ for all $z \in V$. In particular, for ϵ small enough $z_\epsilon \in V$ where

$$z_\epsilon = x_j + \epsilon \cdot \bar{1} = (x_j(\omega_1) + \epsilon, \ldots, x_j(\omega_k) + \epsilon).$$

This is a contradiction with the fact that x_j is the demand. Note that x_j is P_j-measurable so z_ϵ is also P_j-measurable. Indeed, $z_\epsilon \in B_j(p)$ and as $x_j \ll z_\epsilon$, by monotonicity of preferences, $U_j(x_j) < U_j(z_\epsilon)$. Hence, for all $i \in N$ it is satisfied that $\sum_{\omega\in\Omega} p(\omega) \cdot x_i(\omega) = \sum_{\omega\in\Omega} p(\omega) \cdot e_i(\omega)$ and then, by adding the n budget sets, $\sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^{n} x_i(\omega) = \sum_{\omega\in\Omega} p(\omega) \sum_{i=1}^{n} e_i(\omega)$.

As a consequence of these considerations, we remark that if preferences are weakly monotone, Definition 2.1 covers the notion of Radner equilibrium for pure exchange economies with private information.

Note that the above definition allows for free disposal. In the literature the resource feasibility condition in the definition of a Radner equilibrium is written usually with inequality allowing disposability of at least part of the endowments. As pointed out already by Radner (1982, p.945) "the total amount to be disposed may not be measurable with respect to the information of a single trader". On other hand, Einy y Shitovitz (2001) and Glycopantis, Muir and Yannelis (2003) provide examples of economies with differential information without any Radner equilibria with positive prices in the case of non-free disposal. It should be stressed that in these examples free disposal is considered only at those states of nature that no agent can distinguish. Liu (1992) notes that precisely if no consumer can identify a particular state of nature, the aggregate consumption contingent on this state is constrained by this lack of information and, therefore, the demand may be strictly less than the supply in this state.

However, it is not difficult to prove that if each state of nature is distinguished by at least one agent, then any Walrasian expectations equilibrium is with non-free disposal and prices are positive. To formalize this idea we state the following hypothesis:

(H) Given any state $\omega \in \Omega$, there exists an agent who can distinguish it, that is, there exists $i \in N$, of course i depending on ω, such that $P_i(\omega) = \{\omega\}$.

We also establish a strong monotonicity assumption of preferences:

(**SM**) If $x \leq y$ and $x \neq y$ then $U_i(x) < U_i(y)$.

Now we will show that under hypothesis (H) and whenever the preferences are strongly monotone, any Radner equilibrium is a Radner equilibrium with non-free disposal.

Theorem 2.1 *Let E be an information economy satisfying hypothesis (H) and (SM). If (x, p) is a Walrasian expectations equilibrium, then (x, p) is a non-free disposal Walrasian expectations equilibrium, i.e.,* $\sum_{i=1}^{n} x_i(\omega) = \sum_{i=1}^{n} e_i(\omega)$.

Demonstration.

Let (x, p) be a Walrasian expectations equilibrium for an economy E and suppose that $\sum_{i=1}^{n} x_i^m(\omega) < \sum_{i=1}^{n} e_i^m(\omega)$ for a state of nature ω and for a physical commodity m. Then, by Walras' Law, we have that $p^m(\omega) = 0$. By condition (H), it follows that there exists an agent j who distinguishes ω, that is, $P_j(\omega) = \{\omega\}$.

Consider the consumption bundle y identical to x_j except for the commodity m and state ω for which $y^m(\omega) = x_j^m(\omega) + \left(\sum_{i=1}^{n} e_i^m(\omega) - \sum_{i=1}^{n} x_i^m(\omega)\right)$. Observe that y is P_j-measurable. Also, since $p^m(\omega) = 0$, we have that $p \cdot y = p \cdot x_j$ so y is in the budget set. By the strong monotonicity of the utility functions, $U_j(y) > U_j(x_j)$. Then, x_j does not maximize U_j on $B_j(p)$, a contradiction with the fact that (x, p) is a Walrasian expectations equilibrium.

Q.E.D.

Now, given a differential information economy E we associate with it an economy where traders' information is introduced by constraining their consumption set. For each $i \in N$ let denote by $\mathbb{P}_i$ the set of consumption bundles that are P_i-measurable, that is,

$$\mathbb{P}_i = \left\{ x \in (\mathbb{R}^{\ell}_{+})^k \mid x \text{ is } P_i\text{-measurable} \right\}.$$

Let $\mathcal{E}$ denote the Arrow-Debreu economy which coincides with E except for the consumption set of each agent i that now is $\mathbb{P}_i$. Since every agent's initial endowment is P_i-measurable, we have that $e_i \in \mathbb{P}_i$.

The new Arrow-Debreu economy is described by

$$\mathcal{E} = (\mathbb{P}_i, U_i, e_i, i = 1, ..., n),$$

where $\mathbb{P}_i$ is now the consumption set of agent i and, therefore, the utility function U_i can be supposed to be defined over $\mathbb{P}_i$. Observe that in this economy $\mathcal{E}$, due to the information constraints, the consumption sets differ from agent to agent.

Let $\mathbb{P} = \prod_{i=1}^{n} \mathbb{P}_i$. An allocation $x \in \mathbb{P}$ is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} e_i$, (that is, if it is physically feasible in E).

Definition 2.2 *A pair (x, p), where $x = (x_1, ..., x_n) \in \mathbb{P} = \prod_{i=1}^{n} \mathbb{P}_i$ is an allocation and p is a price system, is a competitive equilibrium if*

(i) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i$, *and*

(ii) x_i *maximizes* U_i *on* $B_i(p)$ *for all* $i \in N$.

It is easy to check that, under the hypothesis stated, both economies, E and $\mathcal{E}$, are equivalent with regard to the equilibrium solution. Precisely, the formulation of the consumption sets $\mathbb{P}_i$ guarantees the informational feasibility condition of an allocation in $\mathcal{E}$ as an allocation in E.

3 Walrasian expectations and Nash.

Let $E = \Big((\mathbb{R}^\ell_+)^k, (U_i, e_i, P_i), i = 1, ..., n \Big)$ be the differential information economy described in Section 2. The purpose of this section is to relate Walrasian expectations equilibria to Nash equilibria of a game in normal form.

Given the economy E above, we construct a game in normal form. Let $\Gamma = \{\Theta_i, \pi_i\}_{i=1,\ldots,n}$ be a n-person game where Θ_i is the strategy set and π_i the payoff function of a player i.

The set of strategies Θ_i for a player i is defined by

$$\Theta_i = \Big\{ (x_i, p_i) \in (\mathbb{R}^\ell)^k \times \Delta \mid -\bar{e} \leq x_i \leq \bar{e} \text{ and } p_i \cdot x_i = p_i \cdot e_i \Big\}$$

where $\bar{e} = \sum_{i=1}^{n} e_i$ is the total initial endowment. Therefore, a player i's strategy θ_i is defined by a vector $x_i \in (\mathbb{R}^\ell)^k$ and a price vector $p_i \in \Delta$ such that $p_i \cdot x_i = p_i \cdot e_i$. Let $\Theta = \prod_{i=1}^{n} \Theta_i$ be the set of possible strategy profiles.

Given a strategy profile $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$, where $\theta_i = (x_i, p_i)$, let denote

$$A_i(\theta) = \{ j \in \{1, \ldots, n\} \mid p^j = p^i \}$$

the set of agents who propose the same prices as player i in the strategy profile θ. Agents who announced different price vectors do not trade at all.

Exchange of commodities then takes place among members of $A_i(\theta)$. For every consumer i, we consider the function $g_i : \Theta \rightarrow (\mathbb{R}_+^\ell)^k$ defined by

$$g_i(\theta) = x_i - \frac{\sum_{j \in A_i(\theta)} (x_j - e_j)}{Card\ (A_i(\theta))}$$

where $Card\,(A_i(\theta))$ denotes the cardinality of $A_i(\theta)$.

Note that when the excess of demand is redistributed among the agents who propose the same prices, the resulting outcome that an agent receives may not be informationally compatible with her information. In this case, as the result is not in accordance with her private information, the agent may not be able to understand the outcome and then she rules out the proposed result.

In order to formalize this idea, we define the "outcome function" f_i as

$$f_i(\theta) = \begin{cases} g_i(\theta) & \text{if} \quad g_i(\theta) \in \mathbb{P}_i \\ & \\ e_i/2 & \quad \text{otherwise} \end{cases}$$

The function $f_i : \Theta \rightarrow \mathbb{P}_i \subset (\mathbb{R}_+^\ell)^k$ denotes the commodity bundle that player i receives under each strategy profile $\theta \in \Theta$. Observe that player i receives $g_i(\theta)$ whenever $g_i(\theta)$ is non-negative and compatible with her information. Otherwise, she rules out the bundle and we formalize this by assigning her something that, under monotonicity, she can improve upon.

Then, the i^{th} agent payoff function $\pi_i : \Theta \to \mathbb{R}$ is defined by

$$\pi_i(\theta) = U_i(f_i(\theta)).$$

For a profile θ, let θ_{-i} denote a strategy selection for all players but i. So we write $\theta = (\theta_{-i}, \theta_i)$. A strategy profile $\theta^* = (\theta_1^*, \theta_2^*, ..., \theta_n^*)$ is a **Nash equilibrium** if for all players $i \in N$

$$U_i(f_i(\theta^*_{-i}, \theta^*_i)) \geq U_i(f_i(\theta^*_{-i}, \theta_i)), \text{ for all } \theta_i \in \Theta_i$$

In addition, a strategy profile $\theta^* = (\theta_1^*, \theta_2^*, ..., \theta_n^*)$ is said to be a **strong Nash equilibrium** if it is not upset by any coalition in N, that is, if it is not true that there exist a coalition S and another strategy profile $\check{\theta}$ such that for every $i \in S$

$$U_i(f_i((\theta^*)_{i \notin S}, (\bar{\theta})_{i \in S})) \geq U_i(f_i(\theta^*))$$

with strict inequality for some $i \in S$

In order to establish our main result we state the following assumptions:

- **(E)** (*Strict positivity of endowments*). $e_i \gg 0$ for all $i \in N$.
- **(U)** For every agent i the utility function U_i is strictly monotone in the interior of $(\mathbb{R}^\ell_+)^k$ (i.e., if $0 \ll x \leq y$ and $y \neq x$ then $U_i(y) > U_i(x)$), strictly quasi-concave and differentiable. Moreover, every agent i prefers an interior commodity bundle to any consumption bundle in the frontier of $(\mathbb{R}^\ell_+)^k$.

We remark that the assumptions (H), (E) and (U) allow us to conclude that any equilibrium allocation is strictly positive and there is no free disposal. Let $d_i(p)$ denote the bundle that maximizes U_i on $\{x \in \mathbb{P}_i \mid p \cdot x \leq p \cdot e_i \text{ and } x \leq \bar{e}\}$ (the demand of agent i at prices p). Note that, since U_i is a continuous function, the demand exists. Note also that the strict convexity of preferences guarantees that, given any price system p, the consumption bundle that maximizes U_i on $\{x \in \mathbb{P}_i \mid p \cdot x \leq p \cdot e_i \text{ and } x \leq \bar{e}\}$ is unique.

Theorem 3.1 *Let E be an economy with private information satisfying assumptions (H), (E) and (U), with $n \geq 3$. Let $\Gamma = \{\Theta_i, \pi_i\}_{i=1,\ldots,n}$ be the game in normal form associated with E. Then,*

I. If θ^* is a Nash equilibrium of the game Γ, then $p_i^* = p^*$ for all $i \in N$ and $(f_i(s^*), p^*)$ is a Walrasian expectations equilibrium of the economy E.

II. If (x_i^*, p^*) is a Walrasian expectations equilibrium of E, then $\theta_i^* = (x_i^*, p^*)$ for each $i = 1, \ldots, n$ is a Nash equilibrium of Γ.

Demonstration.

I. Let θ^* be a Nash equilibrium of Γ. We will see that $(f_i(\theta^*), p^*)$ is a Walrasian expectations equilibrium of E.

First we will see that if $\theta^* = (\theta_1^*, \ldots, \theta_n^*)$, with $\theta_i^* = (x_i^*, p_i^*)$, is a Nash equilibrium of Γ, then $g_i(\theta^*) \in \mathbb{P}_i$ for all $i = 1, \ldots, n$. For it suppose that there exists an agent i such that $g_i(\theta^*) \notin \mathbb{P}_i$. Then $f_i(\theta^*) = e_i/2$. Consider that player i chooses $\theta_i^{'} = (e_i, p)$ with $p \neq p_j^*$ for any $j \neq i$. Note that in this case $A_i(\theta') = \{i\}$, so $f_i(\theta^*_{-i}, \theta'_i) = e_i$. Then by monotonicity of U_i, it follows that $U_i(e_i) > U_i(e_i/2)$, that is, $\pi_i(\theta^*_{-i}, \theta'_i) > \pi_i(\theta^*)$, so θ^* would not be a Nash equilibrium of Γ. Hence, if θ^* is a Nash equilibrium of Γ, then $f_i(\theta^*) = g_i(\theta^*)$ for all $i \in N$.

Fact 1. *For any different agents $i, j \in N$, $U_i(f_i(\theta^*)) \geq U_i(d_i(p^*_j))$.*

In order to prove *Fact 1* we will distinguish two cases: (i) $j \in A_i(\theta^*)$ y (ii) $j \notin A_i(\theta^*)$.

If (i) occurs, then $p^*_i = p^*_j$. In this case $Card\,(A_i(\theta^*)) - 1 \neq 0$ because $Card\,(A_i(\theta^*)) \geq 2$. Player i could receive her demand at prices p^*_j by choosing her strategy $\theta_i = (x_i, p^*_j)$, where

$$x_i = \frac{1}{Card\,(A_i(\theta^*)) - 1}\left[d_i(p^*_j) \cdot Card\,(A_i(\theta^*)) + \sum_{r \in A_i(\theta^*) \setminus \{i\}} x^*_r - \sum_{r \in A_i(\theta^*)} e_r\right]$$

Note that $p^*_j \cdot x_i = p^*_j \cdot e_i$ so θ_i is a strategy for agent i, i.e., $\theta_i \in \Theta_i$. Since θ^* is a Nash equilibrium then

$$U_i(f_i(\theta^*_{-i}, \theta^*_i)) \geq U_i(f_i(\theta^*_{-i}, \theta_i)) = U_i(d_i(p^*_j)).$$

In fact, these outcomes not only provide player i the same utility but also are identical. Now we will prove that for all $r \in A_i(\theta^*)$, $f_r(\theta^*) \in B_r(p^*_j)$. Suppose that there exists an agent $r \in A_i(\theta^*)$, such that $p^*_j \cdot f_r(\theta^*) > p^*_j \cdot e_r$. As $\sum\limits_{r \in A_i(\theta^*)} f_r(\theta^*) = \sum\limits_{r \in A_i(\theta^*)} e_r$, then it has to exist another agent $r' \in A_i(\theta^*)$ such that $p^*_j \cdot f_{r'}(\theta^*) < p^*_j \cdot e_{r'}$. By monotonicity of preferences, $U_{r'}(f_{r'}(\theta^*)) < U_{r'}(d_{r'}(p^*_j))$ which is a contradiction. Then $f_r(\theta^*) \in B_r(p^*_j)$. Hence, by the strict quasi-concavity of the utility functions, $f_i(\theta^*) = d_i(p^*_j)$.

If (ii) occurs, then $p_i^* \neq p_j^*$. In this case player i could still receive her demand at prices p_j^* by choosing her strategy $\theta_i = (x_i, p_j^*)$, where

$$x_i = \frac{1}{Card\,(A_j(\theta^*))} \cdot \left[d_i(p_j^*) \cdot (Card\,(A_j(\theta^*)) + 1) + \sum_{r \in A_j(\theta^*)} x_r^* - \sum_{r \in A_j(\theta^*) \bigcup \{i\}} e_r \right]$$

Note that $p_j^* \cdot x_i = p_j^* \cdot e_i$ so $\theta_i \in \Theta_i$. Since θ^* is a Nash equilibrium it follows that

$$U_i(f_i(\theta_{-i}^*, \theta_i^*)) \geq U_i(f_i(\theta_{-i}^*, \theta_i)) = U_i(d_i(p_j^*)).$$

So $U_i(f_i(\theta^*)) \geq U_i(d_i(p_j^*))$.

Fact 2. Now we will see that if θ^* is a Nash equilibrium, then all players propose the same prices:

If there exists an agent i such that $Card\,(A_i(\theta^*)) \geq 2$, then $A_j(\theta^*) = \{1, \ldots, n\}$ for every j.

For it suppose that there exists an agent j such that $j \notin A_i(\theta^*)$, so $p_i^* \neq p_j^*$. Since $\sum_{r \in A_i(\theta^*)} f_r(\theta^*) = \sum_{r \in A_i(\theta^*)} e_r$, there are two cases:

(a) There exists $r \in A_i(\theta^*)$ such that $p_j^* \cdot f_r(\theta^*) < p_j^* \cdot e_r$.

(b) For any $r \in A_i(\theta^*)$, $p_j^* \cdot f_r(\theta^*) = p_j^* \cdot e_r$.

If (a) is the case, there exists $z > 0$ with $z \in \mathbb{P}_r$ such that $p_j^* \cdot (f_r(\theta^*) + z) = p_j^* \cdot e_r$. Since $p \in \Delta$ and by the monotonicity of U_i, it follows that $U_r(d_r(p_j^*)) > U_r(f_r(\theta^*))$, a contradiction with $U_r(f_r(\theta^*)) \geq U_r(d_r(p_j^*))$ above demonstrated.

Suppose that (b) occurs, then $p_j^* \cdot f_r(\theta^*) = p_j^* \cdot e_r$ for any $r \in A_i(\theta^*)$. We will prove that $p_i^* = p_j^*$. Since $r \in A_i(\theta^*)$, it follows by (i) that $f_r(\theta^*) = d_r(p_j^*)$. Given that $Card\, A_i(\theta^*) \geq 2$, $f_r(\theta^*) = d_r(p_i^*)$ for all $r \in A_i(\theta^*)$. Then $d_r(p_j^*) = d_r(p_i^*)$. Since the gradient vector

$$(\partial U_r/\partial x_1 \quad \ldots \quad \partial U_r/\partial x_{l\times k})$$

of the utility function U_r at $d_r(p^*)$ is proportional to p^*, it follows that $p_i^* = p_j^*$ a contradiction with the assumption $p_i^* \neq p_j^*$.

Hence, it only remains to demonstrate that there exists an agent i such that $Card\,(A_i(\theta^*)) \geq 2$ to conclude that in Nash equilibrium all players propose the same prices. For it, and by contradiction, suppose that $A_i(\theta^*) = \{i\}$ for every i. In this case, $f_i(\theta^*) = e_i$ for every i. Consider any agent i. There exists $p \in \Delta$ such that $d_i(p) = e_i$. Since $n \geq 3$, $Card\,\{p_j^* \mid j \neq i\} \geq 2$ and there exists $j \neq i$ such that $p_j^* \neq p$. Then, by definition of p, $U_i(d_i(p_j^*)) > U_i(d_i(p))$. By (ii), $U_i(f_i(\theta^*)) \geq U_i(d_i(p_j^*))$, so $U_i(f_i(\theta^*)) > U_i(e_i)$, a contradiction. Then, $Card\,(A_i(\theta^*)) \geq 2$.

II. Let (x^*, p^*) be a Walrasian expectations equilibrium of E.

Define $\theta_i^* = (x_i^*, p^*)$ for every i. Since $x_i^* \in \mathbb{P}_i$ for every i, it follows that $f_i(\theta^*) = x_i^*$. Furthermore, $x_i^* = d_i(p^*)$.

Let $\tilde{\theta}_i = (x_i^*, \tilde{p})$ with $\tilde{p} \neq p^*$. Then $f_i(\theta_{-i}^*, \tilde{\theta}_i) = e_i$ so $\pi_i(\theta_{-i}^*, \tilde{\theta}_i) = U_i(e_i) \leq \pi_i(\theta^*) = U_i(d_i(p^*))$. On other hand, let $\hat{\theta}_i = (\hat{x}_i, p^*)$. As $f_i(\hat{\theta}_i) \in B_i(p^*)$ then $\pi_i(\theta_{-i}^*, \hat{\theta}_i) = U_i(\hat{x}_i) \leq \pi_i(\theta^*) = U_i(d_i(p^*))$.

Therefore, given the strategy profile θ^*, no agent i can get greater payoffs by choosing a strategy different from θ_i^*, while the other players choose θ_{-i}^*.

Hence, θ^* is a Nash equilibrium of Γ.

Q.E.D.

Finally, we will proof that if θ^* is a Nash equilibrium of Γ, then θ^* is a strong Nash equilibrium of Γ and we state the following corollary.

Corollary 3.1 *Let E be an economy with information satisfying hypothesis (H), (E) and (U) , with $n \geq 3$. Let $\Gamma = \{\Theta_i, \pi_i\}_{i=1,\ldots,n}$ denote the game in normal form associated to E. Then,*

I. *If θ^* is a Nash equilibrium of Γ, then θ^* is a strong Nash equilibrium, $p_i^* = p^*$ for every $i \in N$ and $(f_i(\theta^*), p^*)$ is a Walrasian expectations equilibrium of E.*

II. *If (x_i^*, p^*) is a Walrasian expectations equilibrium of E, then $\theta_i^* = (x_i^*, p^*)$ for every $i = 1, \ldots, n$ is a strong Nash equilibrium of Γ.*

Demonstration.

Let θ^* be a a Nash equilibrium of Γ. By I in Theorem 3.1, every player proposes the same price p^* and $((f_i(\theta^*))_{i\in N}, p^*)$ is a Walrasian expectations equilibrium of the economy. Indeed, as we know, every equilibrium allocation is in the core of the economy.

Now, suppose that θ^* is not a strong Nash equilibrium. Then there exist an arbitrary coalition S and a strategy profile $\bar{\theta}$ such that for all $i \notin S$, $\bar{\theta}_i = \theta_i^*$ and $U_i\left(f_i((\theta^*)_{i\notin S}, (\bar{\theta})_{i\in S})\right) \geq U_i(f_i(\theta^*))$ with strict inequality of at least one i.

So, there exists an agent $i \in S$ such that $U_i\left(f_i((\theta^*)_{i \notin S}, (\bar{\theta})_{i \in S})\right) > U_i(f_i(\theta^*))$. Then $\bar{p_i} \neq p_i^*$ and $f_i(\bar{\theta}) \neq e_i$ which imply that the set $A_i(\bar{\theta})$ of agents that propose the same price has more than one element, i.e., $Card\ (A_i(\bar{\theta})) > 1$. Thus, the coalition $A_i(\bar{\theta})$ would block the outcome $(f_i(\theta^*))_{i \in N}$ so that $\sum\limits_{r \in A_i(\bar{\theta})} f_r(\bar{\theta}) = \sum\limits_{r \in A_i(\bar{\theta})} e_r$ and $U_r(f_r(\theta^*_{-r}, \bar{\theta}_r)) \geq U_k(f_r(\theta^*_{-r}, \theta^*_r))$ for all $r \in A_i(\bar{\theta})$ with strict inequality for at least one i. Therefore, $(f_i(\theta^*))_{i \in N}$ would not be a Walrasian expectations equilibrium of the economy. Hence, we conclude that the strategy profile θ^* is a strong Nash equilibrium.

Q.E.D.

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