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On existence, continuity and utility representation of strictly monotonic preferences *

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Abstract

We consider a continuum of commodities represented by a real interval or, more generally, by a convex set K of any Euclidean space. A consumption plan is a real function defined on the set of commodities K. We show that strictly monotonic preferences on the set of consumption plans always exist. However, they are not representable by utility functions. Moreover, if we consider a consumption set with topological structure, as the positive cone of the Banach space of bounded functions on K, we show that continuous preferences cannot be strictly monotonic.

Keywords: strictly monotonic preferences, representation by utility functions, continuous preferences.

JEL Classification: D11; D50 .

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1 Introduction

In this work we investigate monotonic (more is better) preferences. Our concern is the existence, the representability by utility functions and the continuity of the monotonic preferences.

Let denote K the set of commodities. A consumption plan specifies an amount $f(t) \in \mathbb{R}$ of each commodity $t \in K$. The commodity space is a subspace of the space $\mathbb{R}^K = \{f; f : K \to \mathbb{R}\}$ of the real functions on K. The consumption set or the election set, X, is a subset of the commodity space.

If the set of commodities K is countable, the consumption set X is a subset of the space sequences of real numbers. Moreover, if we consider that consumption plans must be bounded, the election set X will be a subset of l_∞, the Banach space of bounded sequences. Let $\{a_n\}$ be a sequence of strictly positive real numbers such that $\sum a_n < \infty$. Any preference given by the utility function $U(\{x_n\}) = \sum a_n x_n$ is strictly monotone, continuous (with the weak and the norm topology induced on X by the respective topology of l_∞) and, obviously, representable by a utility function.

Thus, in order to consider a large enough set of commodities, we will then suppose that K is a real interval or, more generally, a convex set of any Euclidean space.

We show (Theorem 2) that strictly monotonic preferences always exist. This existence result is a general result that can be applied to any election or consumption set X in which strict monotonicity could be meaningful.

Regarding representability, we show that if the consumption set X contains enough elements (namely a continuum of characteristic functions), any strictly monotonic preference is not representable by utility functions (Theorem 1). The existence result is a value-added result to the non-representability theorem; monotonic preferences always exist and none are representable.

Our proof of non-representability is essentially the same as the classical and well known example of non representability of the lexicographic order. The lexicographic order is, however, difficult to see in economic applications outside the academic textbooks. Therefore, for didactic purposes, strictly monotonic preferences over functions become, in our opinion, a more basic example of a non-representable preference than the lexicographic order.

In regard to the continuous preferences, it is well known that if the consumption set X is endowed with a topology τ and (X, τ) is a connected and separable topological space, then continuous preferences on (X, τ) always have a utility representation (Eilenberg, 1941; Debreu, 1954). Thus, our non-representability result shows that if (X, τ) is a connected and separable topological space, no monotonic preference on (X, τ) is continuous.

In the case of metric spaces, non separability of the metric space X implies the existence of continuous preferences without utility representation (Estévez Toranzo and Hervés-Beloso 1995); however monotonicity could prevent this negative result. In fact, Monteiro, 1987 proves that a continuous preference on a path connected topological space X is representable if and only if it is countably bounded. Then, if X is path connected, any monotonic preference is countably bounded and therefore, any monotonic and continuous preference on X will be representable. This implies that strictly monotonic preferences over a large enough path-connected subset of the space of bounded functions, (as for example the positive cone of the Banach Space of the bounded functions on K) cannot be continuous.

In Section 2 we set some notations, definitions and establish and prove the results. We finish with the conclusion and a final remark.

2 Definitions and Results

Let K be a set of commodities and $X \subset \mathbb{R}^K = \{f; f : K \to R\}$ the consumption set or the set of alternatives for a decision maker. Given $f, g \in X$ we denote $f \geq g$ if $f(t) \geq g(t)$ for all $t \in K$

We will denote a reflexive, complete and transitive preference relation by $\succeq$. The asymmetric part of $\succeq$, will be denoted by $\succ$; Thus $g \succ f$ means that g is more preferred than f.

The preference $\succeq$ defined on X is strictly monotone if, for any f and g in X, with $f \neq g$, we have

$$g \geq f \text{ implies } g \succ f.$$

Let consider the particular case where $X = \{f; f : \mathbb{R} \to \mathbb{R}\} = \mathbb{R}^{\mathbb{R}}$. Next, we will prove that any strictly monotone preference relation $\succeq$ on X is non-representable.

For it, let $f_x = \chi_{[0,x)}$ and $g_x = \chi_{[0,x]}$. If $U : X \to \mathbb{R}$ represents $\succeq$ then $U(f_x) < U(g_x)$ since $g_x \geqslant f_x$. Moreover if $x < y$ then $g_x < f_y$ and therefore $U(g_x) < U(f_y)$. Thus the family of intervals

$$(U(f_x), U(g_x))_{x \geq 0}$$

is pairwise disjoint and therefore uncountable. This is an impossibility that proves our statement.

We have shown that in the space of real functions of a real variable no strictly monotonic preference can be representable by a utility function. This proof is essentially the same as the classical proof of non representability of the lexicographic order. As a first example of non representable preference, for didactic purposes, strictly monotonic preferences over functions become a more basic example of non representability than the lexicographic order.

We remark that the above proof can be adapted to more general sets of functions. If we consider a set X of real functions defined on a set K the same proof applies to X if the following two conditions are fulfilled: i) the domain K of the functions contain two points a and b and the segment $I = \{x \in K; x = \lambda a + (1-\lambda)b\}$, joining these two points, and ii) the set X contain the characteristic functions of the subintervals of I. Thus, we already have proved the following theorem.

Theorem 1 *Let K be any set containing a segment I and let X be any subset of $\mathbb{R}^K$ containing the characteristic functions of the subintervals of I. Then, every strictly monotonic preference on X is non representable by a utility function.*

However we need to show that there exists at least one preference relation on X which is strictly monotone, otherwise this theorem has little meaning.

Theorem 2 *There exists a strictly monotone preference relation on X.*

Proof. Consider the class of strictly monotone preference relations defined on $Y \subset X$. Define a partial order

$$(Y, \succeq_Y) \leq (Z, \succeq_Z) \text{ if } Y \subset Z \text{ and } \succeq_Z \cap Y \times Y = \succeq_Y .$$

Suppose now that $(Y_\alpha, \succeq_\alpha)_\alpha$ is a chain. Let $Y = \cup_\alpha Y_\alpha$ and define

$$x \succeq y \text{ if } x \succeq_\alpha y \text{ for some } \alpha.$$

Alternatively define $\succeq = \cup_\alpha \succeq_\alpha$. Let us show that $\succeq$ is complete: If $f, g \in Y$ is such that $(f,g) \notin \succeq$. Then for every $\alpha, (f,g) \notin \succeq_\alpha$ and therefore $(g,f) \in \succeq_\alpha$ and thus $(g,f) \in \succeq$. Let us show that it is transitive: Suppose $f \succeq g \succeq h$. Then for some α, β we have $f \succeq_\alpha g \succeq_\beta h$. There is a $\gamma \geq \alpha, \beta$. Thus $f \succeq_\gamma g \succeq_\gamma h$ and $f \succeq h$ follows from $f \succeq_\gamma h$. We now show that $\succeq$ is strictly monotone. Suppose $f \geq g, f, g \in Y$. If follows from the monotonicity that $f \succeq g$. Suppose that also $g \succeq f$. Then for some α we have that $g \succeq_\alpha f$ a contradiction with the strict monotonicity for α. Thus every chain has an upper bound. Thus there is a maximal element $(Z, \succeq)$. We now show that $Z = X$. Suppose $Z \neq X$. Take $x \in X \setminus Z$. Let

$$A = \{z \in Z; \exists u \in Z, z \succeq u \text{ and } u \geq x\} \text{ and}$$
$$B = \{z \in Z; \exists v \in Z, x \geq v \succeq z\}.$$

Note that $a \succ b$ if $a \in A$ and $b \in B$ since $a \succeq u \geq x \geq v \succeq b$ and $u \succ v$ by the strong monotonicity. Fix an element $c \in Z \setminus (A \cup B)$. And define $x \sim c$. This completes the extension. If there is no such c we define $A \succ x \succ B$.

■

Remark

Observe that this existence theorem applies to every consumption set X in which strictly monotonicity could be meaningful.

Corollary 3 *In the conditions of Theorem 1, monotonic preferences always exist and no monotonic preference is representable by a utility function.*

The essential requirement of Theorem 1 is a large enough set of commodities. As we have seen in the introduction, if K is countable then strictly monotonic preferences could exist.

The results apply for example, to the case where X is the positive cone of the Banach space *B(K)* of bounded functions on a convex set K of a Euclidian space.

In Monteiro 1987, it is proved that a continuous preference on a path connected topological space X is representable if and only if it is countably bounded. It is easily seen that a strictly monotonic preference on the positive cone $B(K)_+$ is countably bounded. Therefore, we conclude that no continuous preference on $B(K)_+$ is strictly monotonic.

A continuous preference on a compact topological space has a best and worst point. Thus, any continuous preference on a compact or σ-compact (a union of a countable family of compact sets) path connected topological space is representable by utility functions. Therefore, if we consider a topology τ in the consumption set $X \subset \mathbb{R}^K$ such that (X, τ) is a compact or σ-compact path connected topological space, no continuous preference on X is strictly monotonic.

3 Conclusion

Having considered the general scenario where monotonicity is meaningful, we have proved that strictly monotonic preferences always exist.

If we also assume that the set of commodities is large enough, then we show the incompatibility between the continuity and strict monotonicity and between the representation by utility functions and strict monotonicity of the preference relations.

It is important to highlight that our incompatibility results do not apply, for example, to the case of the Banach spaces of the class of integrable functions. In fact,the argument that leads to the proof of our theorem 1, requires that the characteristic functions of the intervals [0,x) and [0,x] be different. However this is not the case if we consider classes of integrable functions.

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