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## A generalization of obligation rules for minimum cost spanning tree problems*

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# A generalization of obligation rules for minimum cost spanning tree problems*

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**Abstract**

Tijs *et al* (2006) introduce the family of obligation rules for minimum cost spanning tree problems. We give a generalization of such family. We prove that our family coincides with the set of rules satisfying an additivity property and a cost monotonicity property. We also provide two new characterizations for the family of obligation rules using the previous properties. In the first one we add a property of separability; and in the second one we add core selection.

**Keywords**: minimum cost spanning tree problems, obligation rules, cost monotonicity, cost additivity, separability, core selection.

## 1 Introduction

A group of agents demand specific services which can only be provided by a common supplier, called the source. Agents can be served through connections to the source, either directly or through other agents and hence a costly activity. These situations are studied in the literature on "minimum cost spanning tree problems", briefly *mcstp*. Formally, an *mcstp* is characterized by a set $N \cup \{0\}$ and a matrix $C$. $N$ is the set of agents, 0 is the source, and for each $i, j \in N \cup \{0\}$, $c_{ij}$ denotes the cost of connecting $i$ and $j$. Many real situations can be modeled in this way. For instance communication networks, such as telephone, internet, wireless telecommunication, or cable television.

Initially, the objective is to minimize the cost of connecting all agents to the source. This is achieved by a network of links that has no cycles, which is called a "minimal cost spanning tree", briefly *mt*. Kruskal (1956) and Prim (1957) designed two algorithms for obtaining an *mt*. Once such a tree is obtained, its associated cost has to be divided among the agents. Some authors propose a single rule for dividing the cost. See, for instance, Bird (1976), Feltkamp *et al* (1994), Kar (2002), and Dutta and Kar (2004).

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Other authors have studied a family of rules instead of focusing on a single rule. In general, each family of rules depends on a family of parameters that model relevant aspects of the *mcstp*, which do not appear in the cost matrix. This freedom allows a planner to choose the rule of the family which best fits a particular *mcstp*, which the planner is trying to solve. For instance, Bergantiños and Lorenzo-Freire (2008a, 2008b) introduce the family of optimistic weighted Shapley rules. Each family rule is a weighted Shapley value of the so called optimistic game (Bergantiños and Vidal-Puga, 2007b). Thus, each rule depends on a vector of weights $(w_i)_{i \in N}$ in such a way that, the larger the weight of an agent is, the more the agent pays. Suppose that the source is a dam which provides water for people in a valley, as in Bergantiños and Lorenzo (2008). Since there are farmers and householders in the valley, agents achieve different benefits from water supply reliability. We take this aspect into account by using an optimistic weighted Shapley rule, where $w_i$ represents the benefits that agent $i$ obtains from the supply of water.

In this paper we study two sets of rules. We introduce the family of generalized obligation rules, which contains the family of obligation rules introduced in Tijs *et al* (2006). We also provide two new characterizations of obligation rules. As a corollary we obtain characterizations of the folk rule introduced in Feltkamp *et al* (1994).

Obligation rules are associated with obligation functions. The idea is as follows. At each stage of Kruskal's algorithm an arc is added to the network. The cost of this arc will be paid by the agents who benefit from adding this arc. Each agent pays the difference between his obligation before the arc is added to the network and after it is added. In an obligation function the obligation of an agent depends only on the agents in the connected component (of the network induced by Kruskal's algorithm) he belongs to.

In this paper we define generalized obligation rules through generalized obligation functions. The idea is to define obligation rules using Kruskal´s algorithm although, applying generalized obligation functions instead of obligation functions. In a generalized obligation function the obligation of an agent depends on the whole partition of the agents defined by the network induced by Kruskal's algorithm, and not only on the element of the partition to which the agent belongs to. Bergantiños *et al* (2010) define a family of rules through Kruskal's algorithm using the so called sharing functions. In Theorem 1 we prove that this family coincides with the family of generalized obligation rules. Moretti *et al* (2009) and Bergantiños and Kar (2010) study two family of rules containing obligation rules. We also prove that generalized obligation rules are unrelated with both families.

One of the most popular approach to the justification of rules is the axiomatic approach. The idea is to characterize a rule or a set of rules through desirable properties. Bergantiños *et al* (2010) prove that the set of rules induced by sharing functions is characterized with Strong Cost Monotonicity $(SCM)$ and Restricted Additivity $(RA)$. $SCM$ says that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off. Namely a rule must be a non-decreasing function on $C$. A rule $f$ satisfies restricted additivity when it is additive in the cost matrix for each pair of "similar" problems. Thus, generalized obligation rules are characterized with $SCM$ and $RA$.

Using the characterization of generalized obligation rules we provide two new characterizations of obligation rules. The first one with $SCM$, $RA$, and core selection, and the second one with $SCM$, $RA$, and separability. Core Selection says that no coalition of agents has incentives to build their own $mt$. It is equivalent to say that the allocation is in the core of the problem. Separability says that if two subsets of agents, $S$ and $N \setminus S$, can connect to the source separately or can connect jointly, and there are no savings when they connect jointly, the agents must pay the same in both circumstances. Besides, if we add the property of symmetry, namely symmetric agents with respect to their connection costs should pay the same, we obtain two new characterizations for the folk rule.

The paper is organized as follows. In Section 2 we introduce *mcstp*. In Section 3 we define generalized obligation rules. In Section 4 we present the characterizations for obligation rules and the folk rule.

## 2 Preliminaries

In this section we introduce minimum cost spanning tree problems and the notation used in the paper.

Let $\mathcal{N} \subset \mathbb{N} = \{1, 2, \ldots\}$ be the set of all possible agents. Given a finite subset $N \subset \mathcal{N}$, an order $\pi$ on $N$ is a bijection $\pi : N \longrightarrow \{1, \ldots, |N|\}$ where, for each $i \in N$, $\pi(i)$ is the position of agent $i$. Let $\Pi^N$ denote the set of all orders on $N$.

For each $S \subset N$, let $\Delta(S) = \left\{ x \in \mathbb{R}^S_+ : \sum_{i \in S} x_i = 1 \right\}$.

Usually, we consider $N = \{1, \ldots, |N|\}$ as the set of agents and 0 as a special element called the *source*. We denote $N_0 = N \cup \{0\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ gives the cost of a direct link between any two nodes. We assume symmetric costs, *i.e.*, for each $i, j \in N_0$, $c_{ij} = c_{ji} \geq 0$ and for each $i \in N_0$, $c_{ii} = 0$.

We denote the set of all cost matrices with agent set $N$ by $\mathcal{C}^N$. Given $C$, $C' \in \mathcal{C}^N$ we say that $C \leq C'$ if for each $i, j \in N_0$, $c_{ij} \leq c'_{ij}$.

A *minimum cost spanning tree problem*, briefly referred to as an *mcstp*, is a pair $(N_0, C)$ where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is a cost matrix. Given an *mcstp* $(N_0, C)$ and $S \subset N$, we denote the restriction of the *mcstp* to $S_0 = S \cup \{0\}$ by $(S_0, C)$.

A *network* $g$ over $N_0$ is a subset of $\{(i, j) : i, j \in N_0, i \neq j\}$. The elements of $g$ are called *arcs*. Since we assume symmetric costs, we work with undirected arcs, *i.e.*, $(i, j) = (j, i)$.

Given a network $g$ and a pair of distinct nodes $i$ and $j$, a *path from $i$ to $j$ in $g$* is a sequence of distinct arcs $g_{ij} = \{(i_{s-1}, i_s)\}_{s=1}^p$ that satisfy $(i_{s-1}, i_s) \in g$ for each $s \in \{1, 2, \ldots, p\}$, $i = i_0$ and $j = i_p$. A *cycle* is a path from $i$ to $i$ different from $(i, i)$. Given $i, j \in N_0$, we say that $i$, *and $j$ are connected in $g$* if there exists a path from $i$ to $j$.

A *tree* is a network such that for each $i \in N$, there is a unique path from $i$ to the source.

We denote the set of all networks over $N_0$ by $\mathcal{G}^N$ and the set of networks over $N_0$ in such a way that every agent in $N$ is connected to the source by $\mathcal{G}_0^N$.

Given a network $g$, let $P(g) = \{S_k(g)\}_{k=1}^{n(g)}$ denote the partition of $N_0$ in connected components induced by $g$. Formally, $P(g)$ is the only partition of $N_0$ satisfying these two properties:

- If $i, j \in S_k(g)$, then $i$ and $j$ are connected in $g$.
- If $i \in S_k(g)$, $j \in S_l(g)$ and $k \neq l$, then $i$ and $j$ are not connected in $g$.

Given a network $g$ and $i \in N_0$, let $S(P(g), i)$ denote the element of $P(g)$ to which $i$ belongs to.

Norde *et al* (2004) prove that every *mcstp* can be written as a non-negative combination of *mcstp* in which the costs of the arcs are 0 or 1. The next lemma states, using our notation, this result in a slightly different but equivalent way.

**Lemma 0**. For each *mcstp* $(N_0, C)$, there exists a family $\{C^q\}_{q=1}^{m(C)}$ of cost matrices and a family $\{x^q\}_{q=1}^{m(C)}$ of non-negative real numbers satisfying three conditions:

(1) $C = \sum_{q=1}^{m(C)} x^q C^q$.

(2) For each $q \in \{1, \ldots, m(C)\}$, there exists a network $g^q$ such that $c^q_{ij} = 1$ if $(i, j) \in g^q$ and $c^q_{ij} = 0$ otherwise.

(3) Let $q \in \{1, \ldots, m(C)\}$ and $\{i, j, k, l\} \subset N_0$. If $c_{ij} \leq c_{kl}$, then $c^q_{ij} \leq c^q_{kl}$.

Given an *mcstp* $(N_0, C)$ and $g \in \mathcal{G}^N$, we define the *cost* associated with $g$ as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimal tree* for $(N_0, C)$, briefly referred to as an *mt*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. An *mt* always exists, although it may not be unique. Given an *mcstp* $(N_0, C)$, $m(N_0, C)$ denotes the cost of any *mt* $t$ in $(N_0, C)$.

After obtaining an *mt*, one of the most important issues addressed in the literature on *mcstp* is how to divide its cost $m(N_0, C)$ among the agents. A *cost allocation rule* is a map $\psi$ that associates with each *mcstp* $(N_0, C)$ a vector $\psi(N_0, C) \in \mathbb{R}^N$ such that $\sum\limits_{i \in N} \psi_i(N_0, C) = m(N_0, C)$. Given an agent $i \in N$, $\psi_i(N_0, C)$ denotes its payment.

Kruskal (1956) defines an algorithm for constructing an *mt*. The idea is quite simple, the *mt* is constructed by sequentially adding arcs with the lowest cost without introducing cycles. Formally, Kruskal's algorithm is defined as follows.

We start with $A^0(C) = \{(i, j) \mid i, j \in N_0, i \neq j\}$ and $g^0(C) = \emptyset$.

Stage 1**:** Take an arc $(i, j) \in A^0(C)$ such that $c_{ij} = \min\limits_{(k,l) \in A^0(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. We have that

$$\begin{aligned} \left(i^1(C), j^1(C)\right) &= (i, j), \\ A^1(C) &= A^0(C) \setminus \{(i, j)\}, \text{ and} \\ g^1(C) &= \left\{\left(i^1(C), j^1(C)\right)\right\}. \end{aligned}$$

Stage $p+1$**.** We have defined the sets $A^p(C)$ and $g^p(C)$. Take an arc $(i, j) \in A^p(C)$ such that $c_{ij} = \min\limits_{(k,l) \in A^p(C)} \{c_{kl}\}$. If there are several arcs satisfying this condition, select just one. Two cases are possible:

1. $g^p(C) \cup \{(i, j)\}$ has a cycle. Go to the beginning of Stage $p+1$ with $A^p(C) = A^p(C) \setminus \{(i, j)\}$ and $g^p(C)$ the same.
2. $g^p(C) \cup \{(i, j)\}$ has no cycles. Take $\left(i^{p+1}(C), j^{p+1}(C)\right) = (i, j)$, $A^{p+1}(C) = A^p(C) \setminus \{(i, j)\}$, and $g^{p+1}(C) = g^p(C) \cup \left\{\left(i^{p+1}(C), j^{p+1}(C)\right)\right\}$. Go to Stage $p+2$**.**

This process is completed in $|N|$ stages. We say that $g^{|N|}(C)$ is a tree obtained following Kruskal's algorithm. Note that this algorithm leads to a tree, but this is not always unique.

When there is no ambiguity, we write $A^p$, $g^p$, and $(i^p, j^p)$ instead of $A^p(C)$, $g^p(C)$, and $(i^p(C), j^p(C))$, respectively.

Bergantiños *et al* (2010) define a family of rules through Kruskal's algorithm. At each step of the algorithm, an arc is added to the network. Once we add the arc, we divide its cost among the agents. Let $\varrho$ be a function specifying the part of the cost paid by each agent. Each agent will pay the sum of the costs paid in each arc selected by Kruskal's algorithm.

Let $P(N_0)$ denote the set of all partitions over $N_0$. Let $P = \{S_0, S_1, \ldots, S_m\}$ be a generic element of $P(N_0)$ such that $0 \in S_0$. Given $P, P' \in P(N_0)$ we say that $P$ is *finer* than $P'$ if for each $S \in P$, there exists $T \in P'$ such that $S \subset T$. Given $P, P' \in P(N_0)$ we say that $P$ is *1-finer* than $P'$ if $P'$ is obtained joining two elements of $P$. Namely, if $P = \{S_0, S_1, \ldots, S_m\}$ and $P$ is 1-finer than $P'$ then, there exist $S_k, S_l \in P$ such that $P' = \{P \setminus \{S_k, S_l\}, S_k \cup S_l\}$.

A *sharing function* $\varrho$ is a function associating with each pair of partitions $(P, P')$ where $P$ is 1-finer than $P'$, a vector $\varrho(P, P') \in \Delta(N)$ satisfying the following *path independence condition.*

Let $P$, $P' \in P(N_0)$ be such that $P$ is finer than $P'$. Assume that $\left\{P_1^1, P_2^1, ..., P_q^1\right\}$ and $\left\{P_1^2, P_2^2, ..., P_q^2\right\}$ are two sequences of partitions satisfying that $P_1^1 = P_1^2 = P$, $P_q^1 = P_q^2 = P'$ and $P_p^i$ is 1-finer than $P_{p+1}^i$ for each $i = 1, 2$ and $p = 1, ..., q-1$. Then, for each $i \in N$,

$$\sum_{p=1}^{q-1} \varrho_i\left(P_p^1, P_{p+1}^1\right) = \sum_{p=1}^{q-1} \varrho_i\left(P_p^2, P_{p+1}^2\right).$$

We can associate with each sharing function $\varrho$ the rule $f^\varrho$ in *mcstp* defined as follows. For each *mcstp* $(N_0, C)$ and each $i \in N$, we define

$$f_i^\varrho(N_0, C) = \sum_{p=1}^{|N|} c_{i^p j^p} \left[\varrho_i\left(P\left(g^{p-1}\right), P\left(g^p\right)\right)\right].$$

Since Kruskal's algorithm can produce several trees, $f^{\varrho}$ could depend on the tree $g^{|N|}$ selected. Bergantiños *et al* (2010) prove that this is not the case. Thus, $f^{\varrho}$ is well defined for each sharing function $\varrho$.

# 3 Generalized obligation rules

Tijs *et al* (2006) introduce obligation rules for *mcstp* through obligation functions. For each obligation function $o$ we can associate an obligation rule $f^o$. We define generalized obligation functions. Applying the same ideas as in Tijs *et al* (2006), for each generalized obligation function $\theta$, we define the rule $f^{\theta}$. The main result of this section says that the set of rules associated with generalized obligation functions coincides with the set of rules associated with sharing functions introduced in Bergantiños *et al* (2010).

Tijs *et al* (2006) define obligation rules through a matrix called the contribution matrix. They also mention that obligation rules can be obtained through Kruskal's algorithm. We present the definition of obligation rules through Kruskal's algorithm in order to adapt it to this paper.

Given $N \subset \mathcal{N}$, an *obligation function* for $N$ is a map $o$ that assigns to each $S \in 2^{N_0} \setminus \{\emptyset\}$ a vector $o(S) \in \mathbb{R}^S$ satisfying the following conditions. For each $S \in 2^{N_0} \setminus \{\emptyset\}$ such that $0 \notin S$, $o(S) \in \Delta(S)$. For each $S \in 2^{N_0} \setminus \{\emptyset\}$ such that $0 \in S$, $o_i(S) = 0$ for each $i \in S$. For each $S, T \in 2^{N_0} \setminus \{\emptyset\}$ with $S \subset T$ and $i \in S$, $o_i(S) \geq o_i(T)$.

We can associate an *obligation rule* $f^o$ with each obligation function $o$ as follows. At each stage of Kruskal's algorithm an arc $(i^p, j^p)$ is added to the network. The cost of this arc will be paid by the agents who benefit from its construction. We compute the amount paid by each agent using the obligation function.

Given an *mcstp* $(N_0, C)$ and $i \in N$,

$$f_i^o(N_0, C) = \sum_{p=1}^{|N|} c_{i^p j^p}(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i)))$$

where $(i^p, j^p)$ and $g^p$ are obtained through Kruskal's algorithm.

Tijs *et al* (2006) prove that $f^o$ is an allocation rule in *mcstp*, *i.e.*, it does not depend on the *mt* chosen by Kruskal's algorithm.

We define a *generalized obligation function* as a map $\theta : P(N_0) \to \mathbb{R}^N$ satisfying three conditions for each $P = \{S_0, S_1, \ldots, S_m\}$:

1. $\theta_i(P) \geq 0$ for each $i \in N$.
2. $\sum_{i \in N} \theta_i(P) = m$.
3. If $P$ is finer than $P'$ then, $\theta_i(P) \geq \theta_i(P')$ for each $i \in N$.

We first prove that obligation functions can be considered as a subset of generalized obligation functions. Given an obligation function $o$, $P \in P(N)$, and $i \in S \in P$, we define $\theta^o : P(N_0) \to \mathbb{R}^N$ such that $\theta_i^o(P) = o_i(S)$.

**Proposition 1**. $\theta^o$ is a generalized obligation function.

**Proof**. We prove that $\theta^o$ satisfies the three conditions of the definition of a generalized obligation function.

1. $\theta_i^o(P) = o_i(S) \geq 0$.

2. Given $P = \{S_0, ..., S_m\} \in P(N_0)$,

$$\sum_{i \in N} \theta_i^o(P) = \sum_{q=0}^{m} \sum_{i \in S_q} \theta_i^o(P) = \sum_{q=0}^{m} \sum_{i \in S_q} o_i(S_q).$$

Since $o(S) \in \Delta(S)$ for each $S \subset N$ and $o_i(S) = 0$ for each $i \in S$ such that $0 \in S$ we conclude that

$$\sum_{q=0}^{m} \sum_{i \in S_q} o_i(S_q) = \sum_{q=1}^{m} 1 = m.$$

3. Consider $P, P' \in P(N_0)$ such that $P$ is finer than $P'$ and $i \in N$. Thus, $\theta_i^o(P) = o_i(S)$ where $i \in S \in P$. Since $P$ is finer that $P'$, there exists $S' \in P'$ such that $i \in S \subset S'$. Therefore, $\theta_i^o(P') = o_i(S')$. Since $o_i(S) \geq o_i(S')$ when $i \in S \subset S' \subset N_0$, we conclude that $\theta_i^o(P) \geq \theta_i^o(P')$. ■

If $\theta^o$ is the generalized obligation function induced by the obligation function $o$, $P \in P(N_0)$ and $i \in S \in P$ then, $\theta_i^o$ only depends on $S$. Nevertheless if $\theta$ is a generalized obligation function, $\theta_i$ depends on $S$ but also on the rest of the agents $(N_0 \backslash S)$. Thus, we can think of obligation functions as the subset of generalized obligation functions where there are no externalities.

We say that $f$ is a *generalized obligation rule* if there exists a generalized obligation function $\theta$ such that for each $(N_0, C)$ and $i \in N$,

$$f_i(N_0, C) = \sum_{p=1}^{|N|} c_{i^p j^p} (\theta_i(P(g^{p-1})) - \theta_i(P(g^p))).$$

In this case we denote $f = f^\theta$ and we say that $f^\theta$ is the generalized obligation rule associated with the generalized obligation function $\theta$.

**Remark 1**. By Proposition 1, if $f^o$ is the obligation rule associated with the obligation function $o$, then $f^o = f^{\theta^o}$. Namely, $f^o$ is the generalized obligation rule associated with the generalized obligation function $\theta^o$. Hence, obligation rules are a subset of generalized obligation rules.

We now prove that the set of generalized obligation rules coincides with the set of rules associated with sharing functions.

**Theorem 1**.

$$\left\{f^\theta : \theta \text{ is a generalized obligation function}\right\} = \{f^\varrho : \varrho \text{ is a sharing function}\}$$

**Proof**. "$\supset$"

Let $f^\varrho$ be such that $\varrho$ is a sharing function.

Let $P = \{S_0, S_1, ..., S_m\} \in P(N_0)$. There exists a sequence $\{P_0, P_1, ..., P_m\} \subset P(N_0)$ such that $P_0 = P$, $P_m = \{N_0\}$, and for each $q = 1, ..., m$, $P_{q-1}$ is 1-finer than $P_q$. Note that this sequence may not be unique. We define

$$\theta(P) = \sum_{q=1}^{m} \varrho(P_{q-1}, P_q).$$

Since $\varrho$ satisfies the path independence condition, $\theta(P)$ does not depend on the sequence $\{P_0, ..., P_m\}$. Thus, $\theta(P)$ is well defined.

We now prove that $\theta$ is a generalized obligation function.

1. Since $\varrho_i(P, P') \geq 0$ for each $P, P' \in P(N_0)$ with $P$ 1-finer than $P'$ and each $i \in N$, we deduce that $\theta_i(P) \geq 0$ for each $i \in N$.

2.

$$\begin{aligned}\sum_{i\in N}\theta_i(P) &= \sum_{i\in N}\sum_{q=1}^{m}\varrho_i\left(P_{q-1},P_q\right)\\ &= \sum_{q=1}^{m}\left(\sum_{i\in N}\varrho_i\left(P_{q-1},P_q\right)\right)\\ &= \sum_{q=1}^{m}1=m.\end{aligned}$$

3. Assume that $P=\{S_0,...,S_m\}$ is finer than $P'=\{S'_0,...,S'_{m'}\}$. Then, $m'<m$ and there exists a sequence $\{P_0,P_1,...,P_m\}\subset P(N_0)$ such that $P_0=P$, $P_m=\{N_0\}$, for each $q=1,...,m$, $P_{q-1}$ is 1-finer than $P_q$, and $P_{m-m'}=P'$. Thus, given $i\in N$,

$$\theta_i(P')=\sum_{q=m-m'+1}^{m}\varrho_i\left(P_{q-1},P_q\right).$$

Now,

$$\theta_i(P)=\sum_{q=1}^{m}\varrho_i\left(P_{q-1},P_q\right)=\sum_{q=1}^{m-m'}\varrho_i\left(P_{q-1},P_q\right)+\theta_i(P').$$

By definition of $\varrho$, $\varrho_i\left(P_{q-1},P_q\right)\geq 0$ for each $q=1,...,m-m'$. Thus, $\theta_i(P)\geq\theta_i(P')$.

We have proved that $\theta$ is a generalized obligation function.
We now prove that $f^\theta=f^\varrho$. By 3 we have that if $P$ is 1-finer than $P'$ and $i\in N$, then

$$\theta_i(P)-\theta_i(P')=\varrho_i\left(P,P'\right).$$

Now it is trivial to see that $f^\theta=f^\varrho$.
" $\subset$ "
Let $f^\theta$ be such that $\theta$ is a generalized obligation function.
Given $P,P'\in P(N_0)$ where $P$ is 1-finer than $P'$, we define

$$\varrho(P,P')=\theta\left(P\right)-\theta\left(P'\right).$$

Next we prove that $\varrho$ is a sharing function:

1. Assume that $P$ is 1-finer than $P'$. We prove that $\varrho(P,P')\in\Delta(N)$.
   (a) Since $P$ is finer than $P'$ and $\theta$ is a generalized obligation function, $\theta(P)\geq\theta(P')$. Hence, $\varrho_i(P,P')\geq 0$ for each $i\in N$.
   (b) Assume that $P=\{S_0,S_1,...,S_m\}$. Thus, $P'=\left\{S'_0,S'_1,...,S'_{m-1}\right\}$. Now

$$\sum_{i\in N}\varrho_i(P,P')=\sum_{i\in N}\theta_i\left(P\right)-\sum_{i\in N}\theta_i\left(P'\right)=m-(m-1)=1.$$

2. We prove that $\varrho$ satisfies the path independence condition. Assume that $\left\{P_1^1,P_2^1,...,P_k^1\right\}$ and $\left\{P_1^2,P_2^2,...,P_k^2\right\}$ are two sequences of partitions satisfying that $P_1^1=P_1^2=P$, $P_k^1=P_k^2=P'$ and $P_q^i$ is 1-finer than $P_{q+1}^i$ for each $i=1,2$ and $q=1,...,k-1$. For each $i\in N$,

$$\begin{aligned}\sum_{q=1}^{k-1}\varrho_i\left(P_q^1,P_{q+1}^1\right) &= \sum_{q=1}^{k-1}\left(\theta_i\left(P_q^1\right)-\theta_i\left(P_{q+1}^1\right)\right)\\ &= \theta_i\left(P_1^1\right)-\theta_i\left(P_k^1\right)\\ &= \theta_i\left(P\right)-\theta_i\left(P'\right).\end{aligned}$$

Analogously, we can prove that

$$\sum_{q=1}^{k-1} \varrho_i \left(P_q^2, P_{q+1}^2\right) = \theta_i(P) - \theta_i(P').$$

We have proved that $\varrho$ is a sharing function. Now it is trivial to see that $f^\theta = f^\varrho$. ■

We end this section by comparing generalized obligation rules with other sets of rules of the literature.

Bergantiños and Lorenzo-Freire (2008a, 2008b) introduce optimistic weighted Shapley rules. They prove that these rules are obligation rules. Thus, they are also generalized obligation rules.

Moretti *et al* (2009) introduce construct and charge ($CC$) rules. $CC$ rules depend on a charge system specifying how to charge agents during the construction of a spanning tree. The charge systems must satisfy three properties: connection, involvement and total aggregation. By Theorem 1, generalized obligation functions can be interpreted in a similar way using the sharing functions. It is trivial to see that sharing functions satisfy total aggregation but fail connection and involvement. Thus, generalized obligation rules can be seen also as a generalization of construct and charge rules, when the order in which we construct the spanning tree is given by Kruskal's algorithm. If the spanning tree is constructed following a different order, then $CC$ rules are, in general, different from generalized obligation rules. As a consequence both sets of rules are unrelated.

Bergantiños and Kar (2010) prove that obligation rules are a subset of the set of marginalistic values of the irreducible form. In general, marginalistic values and generalized obligation rules are unrelated. There exist marginalistic values that are not generalized obligation rules. Let $f$ be a marginalistic value satisfying that $\sum_{i \in N} f_i(N_0, C) \neq m(N_0, C)$ for some $(N_0, C)$. Thus, $f$ cannot be a generalized obligation rule. Also, there exist generalized obligation rules that are not marginalistic values. Let $\theta$ be such that for any $P$, $\theta_i(P) = 0$ when $i \neq j$. Thus, $f_j^\theta(N_0, C) = m(N_0, C)$ and $f_i^\theta(N_0, C) = 0$ when $i \neq j$. Let $(N_0, C)$ be such that $c_{ij} = 1$ for each $i, j \in N_0$. Any marginalistic value $f$ satisfies that $f_i(N_0, C) = 1$ for each $i \in N$.

# 4 The characterizations of obligation rules

Bergantiños *et al* (2010) characterize the set of rules induced by sharing functions, and hence generalized obligation rules, as the set of rules satisfying restricted additivity and strong cost monotonicity. Adding some properties to the ones used in this result, we can obtain characterizations for the family of obligation rules and for the folk rule. We first consider two properties: core selection (the rule is in the core of the problem) and separability (if there are no savings when two groups of agents connect jointly, agents must pay the same when they connect jointly or separately). The main result of this section says that if we add core selection or separability to restricted additivity and strong cost monotonicity, we obtain two characterizations for obligation rules. If we add symmetry to both characterizations of obligation rules we characterize a single rule, the folk rule.

We introduce the properties used to characterize obligation rules.

*Strong Cost Monotonicity (SCM)*: for each pair of *mcstp* $(N_0, C)$ and $(N_0, C')$ such that $C \leq C'$,

$$\psi(N_0, C) \leq \psi(N_0, C').$$

This property implies that if some connection costs increase, no agent ends up better off. It appears, for instance, in Tijs *et al* (2006), Bergantiños and Vidal-Puga (2007a), and Bergantiños and Kar (2010).

Additivity is a standard property and it has been used in many situations. In the case of *mcstp*, additivity says that if we have two *mcstp* $(N_0, C)$ and $(N_0, C')$, then $\psi(N_0, C + C') = \psi(N_0, C) +$

$\psi(N_0, C')$. But the usual additivity property is incompatible with efficiency $\left(\sum_{i\in N} f_i(N_0,C) = m(N_0,C)\right)$, so, no rule satisfies additivity. Bergantiños and Vidal-Puga (2009) introduce the restricted additivity property, which has been used later by Lorenzo and Lorenzo-Freire (2009).

The *mcstp* $(N_0, C)$ and $(N_0, C')$ are *similar* if there exists an *mt* $t = \{(i^0, i)\}_{i\in N}$ in $(N_0, C)$, $(N_0, C')$, and $(N_0, C + C')$ and an order $\pi = (i_1, \ldots, i_n) \in \Pi^N$ such that $c_{i_1^0 i_1} \le c_{i_2^0 i_2} \le \ldots \le c_{i_n^0 i_n}$ and $c'_{i_1^0 i_1} \le c'_{i_2^0 i_2} \le \ldots \le c'_{i_n^0 i_n}$, *i.e.*, the arcs in the *mt* $t$ are ordered in the same way in both problems.

*Restricted Additivity (RA)*: for each pair of similar *mcstp* $(N_0, C)$ and $(N_0, C')$,

$$\psi(N_0, C + C') = \psi(N_0, C) + \psi(N_0, C').$$

*Core Selection* ($CS$): for each *mcstp* $(N_0, C)$ and each $S \subset N$,

$$\sum_{i\in S} f_i(N_0, C) \le m(S_0, C).$$

$CS$ says that no coalition of agents has incentives to build their own *mt*. This is a standard property which has been used in many papers. For instance Bird (1976), Granot and Huberman (1981), and Dutta and Kar (2004).

*Separability* ($SEP$): for each *mcstp* $(N_0, C)$ and each $S \subset N$ such that $m(N_0, C) = m(S_0, C) + m((N \setminus S)_0, C)$,

$$f_i(N_0, C) = \begin{cases} f_i(S_0, C) & \text{when } i \in S \\ f_i((N \setminus S)_0, C) & \text{when } i \in N \setminus S. \end{cases}$$

Two subsets of agents, $S$ and $N \setminus S$, can connect to the source separately or can connect jointly. If there are no savings when they connect jointly, $SEP$ says that agents must pay the same in both circumstances. This property appears in Megiddo (1978), Granot and Huberman (1981), and Bergantiños and Vidal-Puga (2007a).

We now present the characterizations of obligation rules.

**Theorem 2**. ($a$) $f$ satisfies $RA$, $SCM$, and $CS$ if and only if $f$ is an obligation rule.
($b$) $f$ satisfies $RA$, $SCM$, and $SEP$ if and only if $f$ is an obligation rule.

**Proof**. ($a$) We first prove that obligation rules satisfy the three properties. Lorenzo and Lorenzo-Freire (2009) prove that obligation rules satisfy $RA$. Tijs *et al* (2006) prove that obligation rules satisfy $SCM$ and population monotonicity ($PM$). Bergantiños and Vidal-Puga (2007a) prove that population monotonicity implies $CS$.

Assume that $f$ satisfies $RA$, $SCM$, and $CS$. We prove that there exists an obligation function $o$ such that $f = f^o$.

By Theorem 1 in Bergantiños *et al* (2010), there exists a sharing function $\varrho$ such that $f(N_0, C) = f^{\varrho}(N_0, C)$.

Given $P = \{S_0, S_1, \ldots, S_m\} \in P(N_0)$, we define the *mcstp* $(N_0, C^P)$ where $c^P_{ij} = 0$ if $i, j \in S_k$ for any $k \in \{0, 1, \ldots, m\}$ and $c^P_{ij} = 1$ if $i \in S_k$, $j \in S_{k'}$ with $k, k' \in \{0, 1, \ldots, m\}$, $k \ne k'$.

By the proof of Theorem 1 in Bergantiños *et al* (2010), if $P$ is 1-finer than $P'$, then $\varrho(P, P') = f(N_0, C^P) - f(N_0, C^{P'})$.

By Theorem 1, there exists a generalized obligation function $\theta$ such that $f^{\varrho}(N_0, C) = f^{\theta}(N_0, C)$. By the proof of Theorem 1, given $P = \{S_0, ..., S_m\} \in P(N_0)$,

$$\theta(P) = \sum_{q=1}^{m} \varrho(P_{q-1}, P_q)$$

where $P_0 = P$ and $P_m = \{N_0\}$. Thus,

$$\begin{aligned} \theta(P) &= \sum_{q=1}^{m} \left(f(N_0, C^{P_{q-1}}) - f(N_0, C^{P_q})\right) \\ &= f(N_0, C^P) - f\left(N_0, C^{\{N_0\}}\right). \end{aligned}$$

Since $f$ satisfies $RA$ and $SCM$, $f_i\left(N_0, C^{\{N_0\}}\right) = 0$ for each $i \in N$. Thus, $f = f^{\theta}$ where for each $P \in P(N_0)$ and each $i \in N$, $\theta_i(P) = f_i(N_0, C^P)$.

Given $P \in P(N_0)$ with $i \in S \in P$, we define $P^S = \{S, \{j\}_{j \in N_0 \setminus S}\} \in P(N_0)$. Since $P^S$ is finer than $P$, $C^P \leq C^{P^S}$. Since $f$ satisfies $SCM$, $f(N_0, C^P) \leq f(N_0, C^{P^S})$. By $CS$, $\sum_{j \in S} f_j(N_0, C^{P^S}) \leq m(S_0, C^{P^S})$. Thus,

$$0 \leq \sum_{j \in S} f_j(N_0, C^P) \leq \sum_{j \in S} f_j(N_0, C^{P^S}) \leq m(S_0, C^{P^S}) = \begin{cases} 0 & \text{when } 0 \in S \\ 1 & \text{when } 0 \notin S. \end{cases} .$$

When $0 \in S$, $\sum_{j \in S} f_j(N_0, C^P) = \sum_{j \in S} f_j(N_0, C^{P^S}) = 0$. Since $\sum_{j \in N} f_j(N_0, C^P) = m\left(N_0, C^P\right) = m$, $\sum_{j \in S} f_j(N_0, C^P) = \sum_{j \in S} f_j(N_0, C^{P^S}) = 1$ when $0 \notin S$. Since $f(N_0, C^P) \leq f(N_0, C^{P^S})$, $f_i(N_0, C^P) = f_i(N_0, C^{P^S})$.

Let us define the map $o$ that assigns to each $S \in 2^N \setminus \{\emptyset\}$ the vector $o(S) \in \mathbb{R}^S$ such that $o_i(S) = \theta_i(P^S)$ for each $i \in S$. We prove that $o$ is an obligation function.

- $o_i(S) = \theta_i(P^S) \geq 0$ and

$$\sum_{i \in S} \theta_i(P^S) = \sum_{i \in S} f_i\left(N_0, C^{P^S}\right) = \begin{cases} 0 & \text{when } 0 \in S \\ 1 & \text{when } 0 \notin S \end{cases}$$

  Thus, $o(S) \in \Delta(S)$ when $0 \notin S$. Moreover, when $0 \in S$, $o_i(S) = 0$ for each $i \in S$.

- Let $i \in S \subset T$. Clearly, $P^S$ is finer than $P^T$. Therefore, $\theta_i(P^S) \geq \theta_i(P^T)$ and, hence,

$$o_i(S) = \theta_i(P^S) \geq \theta_i(P^T) = o_i(T).$$

We now prove that $f^{\theta} = f^{o}$.

If $i \in S \in P$, then

$$\theta_i(P) = f_i\left(N_0, C^P\right) = f_i\left(N_0, C^{P^S}\right) = \theta_i(P^S) = o_i\left(S\right).$$

Given a partition $P \in P(N_0)$ remember that $S(P, i)$ denotes the element of the partition $P$ to which $i$ belongs to. Thus,

$$\begin{aligned}
f_i^{\theta}(N_0, C) &= \sum_{p=1}^{|N|} c_{i^p j^p} \left(\theta_i(P(g^{p-1})) - \theta_i(P(g^p))\right) \\
&= \sum_{p=1}^{|N|} c_{i^p j^p} \left(f_i\left(N_0, C^{P(g^{p-1})}\right) - f_i\left(N_0, C^{P(g^p)}\right)\right) \\
&= \sum_{p=1}^{|N|} c_{i^p j^p} \left(f_i\left(N_0, C^{P^{S\left(P(g^{p-1}), i\right)}}\right) - f_i\left(N_0, C^{P^{S(P(g^p), i)}}\right)\right) \\
&= \sum_{p=1}^{|N|} c_{i^p j^p} \left(\theta_i\left(P^{S\left(P(g^{p-1}), i\right)}\right) - \theta_i\left(P^{S(P(g^p), i)}\right)\right) \\
&= \sum_{p=1}^{|N|} c_{i^p j^p} \left(o_i(S(P(g^{p-1}), i)) - o_i(S(P(g^p), i))\right) \\
&= f_i^{o}(N_0, C).
\end{aligned}$$

($b$) We know that obligation rules satisfy $RA$, $SCM$, and $PM$. Bergantiños and Vidal-Puga (2007a) prove that $PM$ implies $SEP$.

Consider an allocation rule $f$ satisfying $RA$, $SCM$, and $SEP$. Using similar arguments to those used in ($a$), we can conclude that there exists a generalized obligation function $\theta$ such that $f = f^{\theta}$. Moreover, for each $P \in P(N_0)$, $\theta(P) = f\left(N_0, C^P\right)$.

Given $P = \{S_0, S_1, \ldots, S_m\}$, it is easy to prove that $m(N_0, C^P) = m(S_0, C^P) + \sum\limits_{k=1}^{m} m((S_k)_0, C^P)$.

Let $i \in S_0 \cap N$. Since $f$ satisfies $SEP$, $f_i(N_0, C^P) = f_i(S_0, C^P)$. Therefore, $\sum\limits_{j \in S_0} f_j(N_0, C^P) = \sum\limits_{j \in S_0} f_j(S_0, C^P) = m(S_0, C^P) = 0$.

Let $k \in \{1, ..., m\}$ and $i \in S_k$. Since $f$ satisfies $SEP$, $f_i(N_0, C^P) = f_i((S_k)_0, C^P)$. Therefore, $\sum\limits_{j \in S_k} f_j(N_0, C^P) = \sum\limits_{j \in S_k} f_j((S_k)_0, C^P) = m((S_k)_0, C^P) = 1$.

Using similar arguments to those used in ($a$), it can be proved that the map $o$ assigning to each $S \in 2^N \setminus \{\emptyset\}$ the vector $o(S) = \theta(P^S)$ is an obligation function and $f = f^o$. ■

Let us introduce two properties which will be used later.

A rule $f$ satisfies *Population Monotonicity* ($PM$) if for each $(N_0, C)$, $S \subset T \subset N$, and $i \in N$, we have that $f_i(T_0, C) \leq f_i(S_0, C)$.

A rule $f$ satisfies *Cone-wise positive linearity* ($CPL$) if for each $(N_0, C)$ and $(N_0, C')$ satisfying that there exists an order $\sigma : \{(i,j)\}_{i,j \in N_0, i<j} \to \left\{1, 2, ...., \frac{n(n+1)}{2}\right\}$ such that for each $i, j, k, l \in N_0$ satisfying that $\sigma(i,j) \leq \sigma(k,l)$, then $c_{ij} \leq c_{kl}$ and $c'_{ij} \leq c'_{kl}$, then $f(N_0, C + C') = f(N_0, C) + f(N_0, C')$.

**Remark 2**[1]. The properties used in Theorem 2 are independent. We will do the following:

($i$) We define a rule $f$ which satisfies $RA$ and $SCM$ but fails $CS$ and $SEP$. Thus, $SEP$ is independent of $RA$ and $SCM$ in part ($a$). Moreover, $CS$ is independent of $RA$ and $SCM$ in part ($b$).

Let $f$ be the egalitarian rule, *i.e.*, $f_i(N_0, C) = \dfrac{1}{|N|} m(N_0, C)$ for each $i \in N$.

It is trivial to see that $f$ satisfies $RA$ and $SCM$. Nevertheless, $f$ does not satisfy $SEP$ and $CS$. Let $(N_0, C)$ be such that $N = \{1, 2\}$ and

$$C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

$f$ does not satisfy $CS$ because $f_1(N_0, C) = 1.5 \geq m(\{1\}_0, C) = 1$.

$f$ does not satisfy $SEP$ because $m(N_0, C) = m(\{1\}_0, C) + m(\{2\}_0, C)$ but $f_1(N_0, C) = 1.5 \neq 1 = f_1(\{1\}_0, C)$.

($ii$) We define a rule $f$ which satisfies $RA$ and $SEP$ but fails $SCM$. Thus, $SCM$ is independent of $RA$ and $SEP$ in part ($a$).

Let $u$ be a function assigning to each $S \in 2^{N_0} \setminus \{\emptyset\}$ a vector $u(S) \in R^S$ satisfying the following conditions. For each $S \in 2^{N_0} \setminus \{\emptyset\}$ such that $0 \notin S$, $\sum\limits_{i \in S} u_i(S) = 1$. For each $S \in 2^{N_0} \setminus \{\emptyset\}$ such that $0 \in S$, $u_i(S) = 0$ for each $i \in S$. By convenience we take $u_i(\emptyset) = 0$ for each $i \in N$.

We can associate with each function $u$ a rule $f^u$ as in the case of an obligation rule $f^o$ associated with an obligation function $o$. Namely, given an *mcstp* $(N_0, C)$, let $g^{|N|}$ be a tree obtained applying Kruskal's algorithm to $(N_0, C)$. For each $i \in N$,

$$f_i^u(N_0, C) = \sum_{p=1}^{|N|} c_{i^p j^p} \left( u_i\left(S\left(P\left(g^{p-1}\right), i\right)\right) - u_i\left(S\left(P\left(g^p\right), i\right)\right)\right).$$

[1] A detailed proof of this remark can be found at the Appendix.

Tijs *et al* (2006) prove that obligation rules are well defined. Using arguments similar to those used by them, we can prove that $f^u$ is well defined.

Lorenzo and Lorenzo-Freire (2009) prove that obligation rules satisfy $RA$. Using similar arguments to those used by them, we can prove that $f^u$ satisfies $RA$.

**Claim 1**. $f^u$ satisfies $SEP$ for each $u$. We avoid the proof.

We now prove that $f^u$ does not satisfy $SCM$ for some $u$. We first define $u$. Given $S \subset N$,

$$u_i(S) = \begin{cases} -0.5 & \text{if } S = \{i, j\} \text{ and } i < j \\ 1.5 & \text{if } S = \{i, j\} \text{ and } i > j \\ \frac{1}{|S|} & \text{otherwise.} \end{cases}$$

Note that if $0 \in S$, $u_i(S) = 0$ for each $i \in S \setminus \{0\}$.

Let $(N_0, C^x)$ be such that $N = \{1, 2\}$, $x > 0$, and

$$C^x = \begin{pmatrix} 0 & 10+x & 90 \\ 10+x & 0 & 2 \\ 90 & 2 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} f_1^u(N_0, C^4) &= c_{12}^4(1+0.5) + c_{01}^4(-0.5) = -4 \text{ and} \\ f_1^u(N_0, C^8) &= c_{12}^8(1+0.5) + c_{01}^8(-0.5) = -6. \end{aligned}$$

Since $C^4 \leq C^8$ we have that $f^u$ does not satisfy $SCM$.

$(iii)$ We define a rule $f$ which satisfies $SCM$, $SEP$, and $CS$ but fails $RA$. Thus, $RA$ is independent of $SCM$ and $SEP$ in part $(a)$. Moreover, $RA$ is independent of $SCM$ and $CS$ in part $(b)$.

Bergantiños and Kar (2010) prove that there exists a rule $f$ which satisfies $SCM$ and $PM$ but fails $CPL$. Bergantiños and Vidal-Puga (2009) prove that if a rule satisfies $RA$, then the rule also satisfies $CPL$. Thus, $f$ does not satisfy $RA$.

Bergantiños and Vidal-Puga (2007a) prove that $PM$ implies $SEP$ and $CS$. Thus, $f$ also satisfies $SEP$ and $CS$.

$(iv)$ We define a rule $f$ which satisfies $RA$ and $CS$ but fails $SCM$. Thus, $SCM$ is independent of $RA$ and $CS$ in part $(b)$.

Given an *mcstp* $(N_0, C)$ and an *mt* $t$, Bird (1976) defined the *minimal network* $(N_0, C^t)$ associated with $t$ as follows: $c_{ij}^t = \max_{(k,l)\in g_{ij}} \{c_{kl}\}$, where $g_{ij}$ denotes the unique path in $t$ from $i$ to $j$. It is well known that the minimal network is independent of the *mt* $t$ chosen. Thus, Bergantiños and Vidal-Puga (2007a) define the *irreducible form* $(N_0, C^*)$ of an *mcstp* $(N_0, C)$ as the minimal network $(N_0, C^t)$ associated with some *mt* $t$.

We define a decomposition of $C^*$ in the conditions of Lemma 0. Let us clarify the decomposition in the next example.

$$C^* = \begin{pmatrix} 0 & 4 & 4 \\ 4 & 0 & 2 \\ 4 & 2 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} + (4-2) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We now present the decomposition in general. Let $t = \{(i^0, i)\}$ be an *mt* in $(N_0, C^*)$. Assume, without loss of generality, that $c_{1^01}^* \leq c_{2^02}^* \leq \ldots \leq c_{|N|^0|N|}^*$. Consider $i_0 = 0$ and define

$$\begin{aligned} i_1 &= \max\left\{ i \in N : c_{i^0i}^* = \min_{j\in N} c_{j^0j}^* \right\} \\ i_q &= \max\left\{ i \in N : c_{i^0i}^* = \min_{j\in N, c_{j^0j}^* > c_{i_{q-1}^0 i_{q-1}}^*} c_{j^0j}^* \right\} \text{ for each } q = 2, \ldots, m(C^*). \end{aligned}$$

Since $C^*$ is an irreducible network, there are at most $|N|$ different costs in $C^*$. Thus, $m(C^*) \leq |N|$ and hence, $i_q$ is well defined for each $q = 1, \ldots, m(C^*)$.

We define $x^1 = c^*_{i_1^0 i_1}$, and for each $q = 2, \ldots, m(C^*)$, $x^q = c^*_{i_q^0 i_q} - c^*_{i_{q-1}^0 i_{q-1}}$. Moreover, for each $q = 1, \ldots, m(C^*)$, $C^{*q}$ is given by

$$c_{ij}^{*q} = \begin{cases} 0 & \text{if } c^*_{ij} < c^*_{i_q^0 i_q} \\ 1 & \text{otherwise.} \end{cases}$$

It is trivial to see that $C^* = \sum_{q=1}^{m(C^*)} x^q C^{*q}$ and that the decomposition satisfies the conditions of Lemma 0.

Let $f^o$ be the obligation rule associated with the obligation function

$$o_i(S) = \begin{cases} 1 & \text{if } i = \min_{j \in S}\{j\} \\ 0 & \text{otherwise.} \end{cases}$$

We define $f$ in the following way:

$$f(N_0, C) = \sum_{q=1}^{m(C^*)} x^q f'(N_0, C^{*q})$$

where

$$f'_i(N_0, C^{*q}) = \begin{cases} \dfrac{1}{|N|} & \text{if } c_{ij}^{*q} = 0 \text{ for each } i, j \in N \text{ and } c_{0i}^{*q} = 1 \text{ for each } i \in N, \\ f_i^o(N_0, C^{*q}) & \text{otherwise.} \end{cases}$$

**Claim 2**. $f$ satisfies $CS$. We avoid the proof.

**Claim 3**. $f$ satisfies $RA$. We avoid the proof.

$f$ does not satisfy $SCM$. Let $N = \{1, 2, 3\}$ and $C$ be such that $c_{ij} = 0$ for each $i, j \in N$ and $c_{0i} = 1$ for each $i \in N$. Let $C'$ be such that $c'_{23} = c'_{13} = 1$ and $c'_{ij} = c_{ij}$ otherwise. $C' \geq C$ but $f_2(N_0, C) = \dfrac{1}{3} > 0 = f_2(N_0, C')$. Thus $f$ does not satisfy $SCM$.

Obligation rules have been characterized in Lorenzo and Lorenzo-Freire (2009) and Bergantiños and Kar (2010). Let us compare all these characterizations.

Lorenzo and Lorenzo-Freire (2009) characterize obligation rules with $RA$ and $PM$. Thus, under $RA$, $PM$ is a strong property. If a rule satisfies $PM$ (and $RA$) it also satisfies $SCM$. This result is not true in general, there exist rules satisfying $PM$ but failing $SCM$ (see Bergantiños and Vidal-Puga (2007a)).

Bergantiños and Kar (2010) characterize obligation rules with $SCM$, $PM$, and $CPL$. In Theorem 2 we use $CS$ or $SEP$ instead of $PM$ and $RA$ instead of $CPL$. Thus, in order to obtain a tight characterization of obligation rules, if we weaken $PM$ until $CS$ or $SEP$, then we must strengthen $CPL$ until $RA$. Analogously, if we weaken $RA$ until $CPL$ in our results, then we must strengthen $CS$ or $SEP$ until $PM$.

Feltkamp *et al* (1994) introduce the folk rule in *mcstp*, which has been studied later in Branzei *et al* (2004) and Bergantiños and Vidal-Puga (2007a, 2007b, 2009). As a corollary of Theorem 2 we can give two new axiomatic characterizations of this rule. In order to do so, we need to introduce the property of symmetry.

We say that $i, j \in N$ are *symmetric* if for each $k \in N_0 \setminus \{i, j\}$, $c_{ik} = c_{jk}$.

We say that $f$ satisfies *Symmetry* ($SYM$) if for each *mcstp* $(N_0, C)$ and each pair of symmetric agents $i, j \in N$, $f_i(N_0, C) = f_j(N_0, C)$.

**Corollary 1**. ($a$) The folk rule is the unique rule satisfying $RA$, $SCM$, $SEP$, and $SYM$.
($b$) The folk rule is the unique rule satisfying $RA$, $SCM$, $CS$, and $SYM$.
**Proof**. It is straightforward and we omit it.

The properties used in these characterizations of the folk rule are not independent.

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# 5 Appendix

**Proof of Claim 1**. Consider an $mcstp$ $(N_0, C)$ and $T \subset N$ such that $m(N_0, C) = m(T_0, C) + m((N \setminus T)_0, C)$.

Let $t^T$ be an $mt$ in $(T_0, C)$ and let $t^{N\setminus T}$ be an $mt$ in $((N \setminus T)_0, C)$. It is trivial to see that $t^t \cup t^{N\setminus T}$ is an $mt$ in $(N_0, C)$.

For each $T' \subset N$, let $(i^p(T'), j^p(T'))$ denote the arc selected at Stage $p$ of Kruskal's algorithm when applied to $(T'_0, C)$.

We can take $g^{|N|} = \{(i^p(N), j^p(N))\}_{p=1}^{|N|}$ such that:

- $g^{|N|} = t^T \cup t^{N\setminus T}$.
- The order in which we select the arcs, following Kruskal's algorithm, is the same in $(T_0, C)$, $((N \setminus T)_0, C)$, and $(N_0, C)$. Namely, given $T' \in \{T, N \setminus T\}$ and $1 \leq p < q \leq |T'|$ such that $(i^p(T'), j^p(T')) = \left(i^{p'}(N), j^{p'}(N)\right)$ and $(i^q(T'), j^q(T')) = \left(i^{q'}(N), j^{q'}(N)\right)$, then $p' < q'$.

We now prove that for each arc $(i^*, j^*) \in t^T \cup t^{N\setminus T}$ the way in which its cost $c_{i^*j^*}$ is divided among the agents is the same when $(i^*, j^*)$ is selected by Kruskal's algorithm applied to $(N_0, C)$ than when $(i^*, j^*)$ is selected in Kruskal's algorithm applied to $(T_0, C)$ when $(i^*, j^*) \in t^T$, or applied to $((N \setminus T)_0, C)$ when $(i^*, j^*) \in t^{N\setminus T}$.

Let $(i^*, j^*) \in t^T$ (the case $(i^*, j^*) \in t^{N\setminus T}$ is similar and we omit it). Thus, $(i^*, j^*) = (i^p(T), j^p(T)) = (i^{p'}(N), j^{p'}(N))$ where $p' \geq p$.

For each $q = 1, \ldots, |T|$, $g^q(T)$ denotes the network obtained at Stage $q$ of Kruskal's algorithm applied to $(T_0, C)$. For each $q = 1, \ldots, |N|$, $g^q$ denotes the network obtained at Stage $q$ of Kruskal's algorithm applied to $(N_0, C)$.

We prove that for each $i \in N$,

$$u_i\left(S\left(P\left(g^{p'-1}\right), i\right)\right) - u_i\left(S\left(P\left(g^{p'}\right), i\right)\right) =$$

$$u_i\left(S\left(P\left(g^{p-1}(T)\right), i\right)\right) - u_i\left(S\left(P\left(g^{p}(T)\right), i\right)\right).$$

If $i \in N \setminus T$, $u_i\left(S\left(P\left(g^{p-1}(T)\right), i\right)\right) - u_i\left(S\left(P\left(g^{p}(T)\right), i\right)\right) = u_i(\emptyset) - u_i(\emptyset) = 0$. Note that agents in $N \setminus T$ pay nothing in $(T_0, C)$.

We know that $g^{p'}$ is obtained from $g^{p'-1}$ joining $S\left(P\left(g^{p'-1}\right), i^*\right)$ and $S\left(P\left(g^{p'-1}\right), j^*\right)$. Moreover, $g^p(T)$ is obtained from $g^{p-1}(T)$ joining $S\left(P\left(g^{p-1}(T)\right), i^*\right)$ and $S\left(P\left(g^{p-1}(T)\right), j^*\right)$.

We consider several cases:

1. $i \notin S\left(P\left(g^{p-1}(T)\right), i^*\right) \cup S\left(P\left(g^{p-1}(T)\right), j^*\right)$.

   If $i \in T$, then $S\left(P\left(g^{p-1}(T)\right), i\right) = S\left(P\left(g^{p}(T)\right), i\right)$. Hence

   $$u_i\left(S\left(P\left(g^{p-1}(T)\right), i\right)\right) - u_i\left(S\left(P\left(g^{p}(T)\right), i\right)\right) = 0.$$

   If $i \in N \setminus T$,

   $$u_i\left(S\left(P\left(g^{p-1}(T)\right), i\right)\right) - u_i\left(S\left(P\left(g^{p}(T)\right), i\right)\right) = u_i(\emptyset) - u_i(\emptyset) = 0.$$

   If $i \in N$, then $S\left(P\left(g^{p'-1}\right), i\right) = S\left(P\left(g^{p'}\right), i\right)$. Hence,

   $$u_i\left(S\left(P\left(g^{p'-1}\right), i\right)\right) - u_i\left(S\left(P\left(g^{p'}\right), i\right)\right) = 0.$$

2. $i \in S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right)$ and $0 \notin S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right)$. Thus, $i \in T$. Assume, without loss of generality, that $i \in S\left(P\left(g^{p-1}(T)\right), i^{*}\right)$. Therefore,

$$\begin{aligned} S\left(P\left(g^{p-1}(T)\right), i\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) = S\left(P\left(g^{p'-1}\right), i^{*}\right) \\ &= S\left(P\left(g^{p'-1}\right), i\right), \text{ and} \\ S\left(P\left(g^{p}(T)\right), i\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right) \\ &= S\left(P\left(g^{p'}\right), i\right). \end{aligned}$$

Hence, the result holds.

3. $i \in S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right)$ and $0 \in S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right)$. Thus, $i \in T$. Assume, without loss of generality, that $i \in S\left(P\left(g^{p-1}(T)\right), i^{*}\right)$. We consider two cases:

(a) $0 \in S\left(P\left(g^{p-1}(T)\right), i^{*}\right)$. Thus, $\{0, i\} \subset S\left(P\left(g^{p}(T)\right), i^{*}\right)$. Therefore,

$$\begin{aligned} S\left(P\left(g^{p-1}(T)\right), i\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \text{ and} \\ S\left(P\left(g^{p}(T)\right), i\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right). \end{aligned}$$

Hence,

$$u_i\left(S\left(P\left(g^{p-1}(T)\right), i\right)\right) = u_i\left(S\left(P\left(g^{p}(T)\right), i\right)\right) = 0.$$

Moreover, $\{0, i\} \subset S\left(P\left(g^{p'-1}\right), i^{*}\right) \subset S\left(P\left(g^{p'}\right), i^{*}\right)$. Therefore,

$$\begin{aligned} S\left(P\left(g^{p'-1}\right), i\right) &= S\left(P\left(g^{p'-1}\right), i^{*}\right) \text{ and} \\ S\left(P\left(g^{p'}\right), i\right) &= S\left(P\left(g^{p'}\right), i^{*}\right). \end{aligned}$$

Hence,

$$u_i\left(S\left(P\left(g^{p'-1}\right), i^{*}\right)\right) = u_i\left(S\left(P\left(g^{p'}\right), i^{*}\right)\right) = 0.$$

Thus, the result holds.

(b) $0 \in S\left(P\left(g^{p-1}(T)\right), j^{*}\right)$. Then,

$$\begin{aligned} S\left(P\left(g^{p-1}(T)\right), i\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \text{ and} \\ 0 \in S\left(P\left(g^{p}(T)\right), i\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \cup S\left(P\left(g^{p-1}(T)\right), j^{*}\right). \end{aligned}$$

Hence,

$$u_i\left(S\left(P\left(g^{p-1}(T)\right), i\right)\right) - u_i\left(S\left(P\left(g^{p}(T)\right), i\right)\right) = u_i\left(S\left(P\left(g^{p-1}(T)\right), i^{*}\right)\right).$$

Moreover, $S\left(P\left(g^{p'-1}\right), i^{*}\right) = S\left(P\left(g^{p-1}(T)\right), i^{*}\right)$ and $0 \in S\left(P\left(g^{p'-1}\right), j^{*}\right)$. Thus,

$$\begin{aligned} S\left(P\left(g^{p'-1}\right), i^{*}\right) &= S\left(P\left(g^{p-1}(T)\right), i^{*}\right) \text{ and} \\ 0 \in S\left(P\left(g^{p'}\right), i\right) &= S\left(P\left(g^{p'}\right), i^{*}\right). \end{aligned}$$

Hence,

$$u_i\left(S\left(P\left(g^{p'-1}\right), i\right)\right) - u_i\left(S\left(P\left(g^{p'}\right), i\right)\right) = u_i\left(S\left(P\left(g^{p-1}(T)\right), i^{*}\right)\right).$$

This finishes the proof of Claim 1. ■

**Proof of Claim 2**.

A cooperative game with transferable utility, *TU game*, is a pair $(N, v)$ where $N \subset \mathcal{N}$ and $v : 2^N \to \mathbb{R}$ satisfies that $v(\emptyset) = 0$. We denote by $core(N, v)$ the core of $(N, v)$. Since we are allocating costs the core is defined as

$$core(N, v) = \left\{ (x_i)_{i \in N} : \sum_{i \in N} x_i = v(N) \text{ and } \forall S \subset N, \sum_{i \in S} x_i \leq v(S) \right\}.$$

Bird (1976) associates a *TU* game $(N, v_C)$ with each *mcstp* $(N_0, C)$. For each coalition $S \subset N$, the value of a coalition is the cost of connecting agents in $S$ to the source by themselves, *i.e.*, $v_C(S) = m(S_0, C)$. Because of the definition of $CS$, a rule $f$ satisfies $CS$ if an only if $f(N_0, C) \in core(N, v_C)$.

Let $(N_0, C)$ be an *mcstp*. Then, $f(N_0, C) = \sum\limits_{q=1}^{m(C^*)} x^q f(N_0, C^{*q})$ where $C^* = \sum\limits_{q=1}^{m(C^*)} x^q C^{*q}$.

Two cases are possible:

1. Assume that $c_{ij}^{*q} = 0$ for each $i, j \in N$ and $c_{0i}^{*q} = 1$ for each $i \in N$. Thus, for each $i \in N$, $f_i(N_0, C^{*q}) = \frac{1}{|N|}$. Moreover, for each $S \subset N$, $v_{C^{*q}}(S) = 1$. Thus, $f(N_0, C^{*q}) \in core\,(N, v_{C^{*q}})$.
2. Otherwise $f(N_0, C^{*q}) = f^o(N_0, C^{*q})$. By Theorem 2 $(a)$, $f^o$ satisfies $CS$. Hence, $f(N_0, C^{*q}) \in core\,(N, v_{C^{*q}})$.

Now, $f(N_0, C) = \sum\limits_{q=1}^{m(C^*)} x^q f(N_0, C^{*q}) \in core \left( N, \sum\limits_{q=1}^{m(C^*)} x^q v_{C^{*q}} \right)$.

Bergantiños and Vidal-Puga (2009) prove that if $C, C'$, and $C + C'$ satisfy the conditions of the definition of $RA$, then $v_{(C+C')^*} = v_{C^*} + v_{C'^*}$. Thus,

$$\sum_{q=1}^{m(C^*)} x^q v_{C^{*q}} = v_{\sum\limits_{q=1}^{m(C^*)} x^q C^{*q}} = v_{C^*}.$$

Hence, $f(N_0, C) \in core\,(N, v_{C^*})$. Bird (1976) proves that $core\,(N, v_{C^*}) \subset core\,(N, v_C)$. Therefore, $f(N_0, C) \in core(N, v_C)$, which means that $f$ satisfies $CS$. ■

**Proof of Claim 3**.

Let $(N_0, C), (N_0, C')$ and $(N_0, C + C')$ in the conditions of the definition of $RA$. Bergantiños and Vidal-Puga (2009) prove that $(C + C')^* = C^* + C'^*$. A matrix $C$ is irreducible if $C = C^*$. Since $f$ only depends on the irreducible matrix, we assume that $C$, $C'$, and $C + C'$ are irreducible.

Bergantiños and Vidal-Puga (2007a) prove that $(N_0, C)$ is irreducible if and only if there exists an *mt* $t^\pi$ in $(N_0, C)$ satisfying two conditions:

$(A1)$ $t^\pi = \{(\pi_{s-1}, \pi_s)\}_{s=1}^{|N|}$ in $(N_0, C)$ with $\pi_0 = 0$.

$(A2)$ Given $\pi_p, \pi_q \in N_0$ with $p < q$, $c_{\pi_p \pi_q} = \max\limits_{p < s \leq q} \{c_{\pi_{s-1} \pi_s}\}$.

Given an *mt* $t$ in an irreducible problem $(N_0, C)$, Bergantiños and Vidal-Puga (2007a) define a procedure for associating an *mt* $t^\pi$ satisfying $(A1)$ and $(A2)$.

Let $t = \{(i^0, i)\}_{i \in N}$ be the *mt* for the three *mcstp* given by the definition of $RA$. Since $t$ is an *mt* in $(N_0, C), (N_0, C')$ and $(N_0, C + C')$, applying the procedure of Bergantiños and Vidal-Puga (2007a) it is straightforward to prove that there exists an *mt* $t^\pi$ in $(N_0, C), (N_0, C')$ and $(N_0, C + C')$ satisfying $(A1)$ and $(A2)$. Moreover, we can order the cost of the arcs of $t^\pi$ in the same way in the three problems. Namely, there exists an order $\pi'$ of the agents satisfying that if $1 \leq p' < q' \leq |N|$, $\pi'_{p'} = \pi_p$, and $\pi'_{q'} = \pi_q$, then $c_{\pi_{p-1} \pi_p} \leq c_{\pi_{q-1} \pi_q}$ and $c'_{\pi_{p-1} \pi_p} \leq c'_{\pi_{q-1} \pi_q}$.

We say that an *mcstp* $(N_0, C)$ satisfies the property $PRO$ if in the decomposition of $f$ there is no $C^{*q}$ satisfying that $c_{ij}^{*q} = 0$ for each $i, j \in N$ and $c_{0i}^{*q} = 1$ for each $i \in N$.

We prove that $f(N_0, C) + f(N_0, C') = f(N_0, C + C')$. We distinguish three cases:

1. $c_{0\pi_1} \leq \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\}$ and $c'_{0\pi_1} \leq \max_{2<s\leq|N|} \left\{c'_{\pi_{s-1}\pi_s}\right\}$.

   Thus, $(N_0, C), (N_0, C')$ and $(N_0, C + C')$ satisfies $PRO$. In this case we know that $f(N_0, C) = f^o(N_0, C)$, $f(N_0, C') = f^o(N_0, C')$, and $f(N_0, C + C') = f^o(N_0, C + C')$. Since all the problems obtained in the decomposition of $(N_0, C), (N_0, C')$ and $(N_0, C + C')$ satisfy the conditions of the definition of $RA$ and $f^o$ satisfies $RA$,

$$f^o(N_0, C) + f^o(N_0, C') = f^o(N_0, C + C').$$

2. $c_{0\pi_1} > \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\}$ and $c'_{0\pi_1} \leq \max_{2<s\leq|N|} \left\{c'_{\pi_{s-1}\pi_s}\right\}$ (the case $c_{0\pi_1} \leq \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\}$ and $c'_{0\pi_1} > \max_{2<s\leq|N|} \left\{c'_{\pi_{s-1}\pi_s}\right\}$ is similar and so we omit it).

   Thus, $(N_0, C)$ does not satisfy $PRO$ and $(N_0, C')$ satisfies $PRO$. Because of the existence of the order $\pi'$, we deduce that $c_{0\pi_1} + c'_{0\pi_1} > \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\} + \max_{2<s\leq|N|} \left\{c'_{\pi_{s-1}\pi_s}\right\}$. Thus, $(N_0, C + C')$ does not satisfy $PRO$.

   Let us define $\widetilde{C}$ as:

$$\widetilde{c}_{ij} = \begin{cases} 1 & \text{when } 0 \in \{i, j\} \\ 0 & \text{otherwise.} \end{cases} \quad \text{for each } i, j \in N_0.$$

   Applying the decomposition to $C$,

$$C = \sum_{q=1}^{m(C)} x^q C^q = \sum_{q=1}^{m(C)-1} x^q C^q + x^{m(C)} \widetilde{C},$$

   where $x^{m(C)} = c_{0\pi_1} - \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\}$ and $\sum_{q=1}^{m(C)-1} x^q C^q$ satisfies $PRO$.

   Analogously, since $(N_0, C + C')$ does not satisfy $PRO$,

$$C + C' = \sum_{q=1}^{m(C+C')-1} x^{*q} (C + C')^q + x^{*m(C+C')} \widetilde{C},$$

   where $\sum_{q=1}^{m(C+C')-1} x^{*q} (C + C')^q$ satisfies $PRO$.

   Moreover,

$$x^{*m(C+C')} = \left(c_{0\pi_1} + c'_{0\pi_1}\right) - \max_{2<s\leq|N|} \left\{c_{\pi_{s-1}\pi_s} + c'_{\pi_{s-1}\pi_s}\right\}.$$

   Because of the existence of the order $\pi'$,

$$\begin{aligned} \max_{2<s\leq|N|} \left\{c_{\pi_{s-1}\pi_s} + c'_{\pi_{s-1}\pi_s}\right\} &= \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\} + \max_{2<s\leq|N|} \left\{c'_{\pi_{s-1}\pi_s}\right\} \text{ and} \\ \max_{2<s\leq|N|} \left\{c'_{\pi_{s-1}\pi_s}\right\} &= c'_{0\pi_1}. \end{aligned}$$

   Hence, $x^{*m(C+C')} = c_{0\pi_1} - \max_{2<s\leq|N|} \{c_{\pi_{s-1}\pi_s}\} = x^{m(C)}$.

   Therefore,

$$C + C' = \sum_{q=1}^{m(C)-1} x^q C^q + x^{m(C)} \widetilde{C} + C' = \sum_{q=1}^{m(C+C')-1} x^{*q} (C + C')^q + x^{m(C)} \widetilde{C}.$$

Then,

$$\sum_{q=1}^{m(C)-1} x^q C^q + C' = \sum_{q=1}^{m(C+C')-1} x^{*q}(C+C')^q.$$

Since all the problems obtained in the decomposition of $(N_0, C), (N_0, C')$ and $(N_0, C+C')$ satisfy the conditions of the definition of $RA$ and $f^o$ satisfies $RA$,

$$\begin{aligned} f(N_0, C) + f(N_0, C') &= f^o\left(N_0, \sum_{q=1}^{m(C)-1} x^q C^q\right) + x^{m(C)}\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) + f^o(N_0, C') \\ &= f^o\left(N_0, \sum_{q=1}^{m(C)-1} x^q C^q + C'\right) + x^{m(C)}\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) \\ &= f^o\left(N_0, \sum_{q=1}^{m(C+C')-1} x^{*q}(C+C')^q\right) + x^{m(C)}\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) \\ &= f(N_0, C+C'). \end{aligned}$$

3. $c_{0\pi_1} > \max_{2<s\leq|N|}\left\{c_{\pi_{s-1}\pi_s}\right\}$ and $c'_{0\pi_1} > \max_{2<s\leq|N|}\left\{c'_{\pi_{s-1}\pi_s}\right\}$.

   Thus, $(N_0, C)$, $(N_0, C')$, and $(N_0, C+C')$ do not satisfy $PRO$.

   Using similar arguments as in Case 2 we can prove that

$$\begin{aligned} C &= \sum_{q=1}^{m(C)-1} x^q C^q + x^{m(C)}\widetilde{C}, \\ C' &= \sum_{q=1}^{m(C')-1} x'^q C'^q + x'^{m(C')}\widetilde{C}, \text{ and} \\ C + C' &= \sum_{q=1}^{m(C+C')-1} x^{*q}(C+C')^q + x^{*m(C+C')}\widetilde{C} \\ &= \sum_{q=1}^{m(C)-1} x^q C^q + \sum_{q=1}^{m(C')-1} x'^q C'^q + (x^{m(C)} + x'^{m(C')})\widetilde{C}. \end{aligned}$$

   where $\left(N_0, \sum_{q=1}^{m(C)-1} x^q C^q\right)$, $\left(N_0, \sum_{q=1}^{m(C')-1} x'^q C'^q\right)$, and $\left(N_0, \sum_{q=1}^{m(C+C')-1} x^{*q}(C+C')^q\right)$ satisfy $PRO$.

   Moreover, $x^{m(C)} = c_{0\pi_1} - \max_{2<s\leq|N|}\left\{c_{\pi_{s-1}\pi_s}\right\}$, $x'^{m(C')} = c'_{0\pi_1} - \max_{2<s\leq|N|}\left\{c'_{\pi_{s-1}\pi_s}\right\}$, and $x^{*m(C+C')} = (c_{0\pi_1} + c'_{0\pi_1}) - \left(\max_{2<s\leq|N|}\left\{c_{\pi_{s-1}\pi_s} + c'_{\pi_{s-1}\pi_s}\right\}\right)$.

   Because of the existence of the order $\pi'$,

$$\max_{2<s\leq|N|}\left\{c_{\pi_{s-1}\pi_s} + c'_{\pi_{s-1}\pi_s}\right\} = \max_{2<s\leq|N|}\left\{c_{\pi_{s-1}\pi_s}\right\} + \max_{2<s\leq|N|}\left\{c'_{\pi_{s-1}\pi_s}\right\},$$

   and hence, $x^{*m(C+C')} = x^{m(C)} + x'^{m(C')}$.

   Then,

$$\sum_{q=1}^{m(C+C')-1} x^{*q}(C+C)^q = \sum_{q=1}^{m(C)-1} x^q C^q + \sum_{q=1}^{m(C')-1} x'^q C'^q.$$

Since all the problems obtained in the decomposition of $(N_0, C), (N_0, C')$ and $(N_0, C{+}C')$ satisfy the conditions of the definition of $RA$ and $f^o$ satisfies $RA$,

$$\begin{aligned}
f(N_0, C) + f(N_0, C') &= f^o\left(N_0, \sum_{q=1}^{m(C)-1} x^q C^q\right) + x^{m(C)}\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) \\
&\quad + f^o\left(N_0, \sum_{q=1}^{m(C')-1} x'^q C'^q\right) + x'^{m(C')}\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) \\
&= f^o\left(N_0, \sum_{q=1}^{m(C)-1} x^q C^q + \sum_{q=1}^{m(C')-1} x'^q C'^q\right) \\
&\quad + (x^{m(C)} + x'^{m(C')})\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) \\
&= f^o\left(N_0, \sum_{q=1}^{m(C+C')-1} x^{*q}(C + C')^q\right) \\
&\quad + x^{*m(C+C')}\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right) \\
&= f(N_0, C + C'). \blacksquare
\end{aligned}$$