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## An axiomatic approach in minimum cost spanning tree problems with groups

**Gustavo Bergantiños and**
**María Gómez-Rúa**

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Facultade de Ciencias Económicas e Empresariais, Campus As Lagoas-Marcosende, 36310 Vigo.

# An axiomatic approach in minimum cost spanning tree problems with groups

Gustavo Bergantiños, Research Group in Economic Analysis Universidade de Vigo (Spain).*

María Gómez-Rúa, Research Group in Economic Analysis Universidade de Vigo (Spain)

**Abstract**

We study minimum cost spanning tree problems with groups, where agents are located in different villages, cities, etc. The groups are the agents of the same village. In Bergantiños and Gómez-Rúa (2010, Economic Theory) we define the rule $F$ as the Owen value of the irreducible game with groups and we prove that $F$ generalizes the folk rule of minimum cost spanning tree problems. Bergantiños and Vidal-Puga (2007, Journal of Economic Theory) give two characterizations of the folk rule. In the first one they characterize it as the unique rule satisfying cost monotonicity, population monotonicity and equal share of extra costs. In the second characterization of the folk rule they replace cost monotonicity by independence of irrelevant trees and population monotonicity by separability. In this paper we extend such characterizations to our setting. Some of the properties are the same (cost monotonicity and independence of irrelevant trees) and the other need to be adapted. In general, we do it by claiming the property twice: once among the groups and the other among the agents inside the same group.

**Keywords:** minimum cost spanning tree problems, folk rule, cost sharing, axiomatization.

*Corresponding author: Tel: +34986812497; fax: +34986812401. E-mail: gbergant@uvigo.es

# 1 Introduction

In minimum cost spanning tree problems, *mcstp*, a group of agents want to be connected to a single supplier of some service. Usually, agents can reduce the total cost if some agents connect to the supplier through other agents. The cheapest graph connecting all agents to the supplier is called the minimum cost spanning tree. It is assumed that agents construct a minimum cost spanning tree. Now a cost allocation problem arises. The most relevant question is how to divide the cost of the minimum cost spanning tree among the agents.

The most popular way to answer this question is through the axiomatic characterization of a single rule (see for instance, Kar (2002), Dutta and Kar (2004), and Branzei *et al*. (2004)) or a family of rules (see for instance, Bergantiños *et al*. (2009), Bogomolnaia and Moulin (2008), and Chun and Lee (2009)).

Sometimes agents are located in different villages (see for instance, Dutta and Kar (2004) or Bergantiños and Lorenzo (2004, 2005, 2008)). This means, in terms of the cost matrix, that the connection cost between two agents of the same village is not larger than the connection cost between an agent of this village and an agent from other village. The classical model of *mcstp* includes these situations. Nevertheless, it ignores the fact that some group of agents are located in the same village. In Bergantiños and Gómez-Rúa (2010) we introduce a new class of problems which includes this fact in the model. We do it by considering an extra element in the model, namely a partition $G = \{G^1, ..., G^m\}$ of the set of agents $N$. For each $k = 1, ..., m$, $G^k$ represents the group of agents located in the same village, city, ... We call these problems *mcstp* with groups.

A very well-known rule in *mcstp* is the folk rule. It was introduced in Feltkamp *et al*. (1994) and studied later in further papers, for instance Branzei *et al*. (2004) and Bergantiños and Vidal-Puga (2007a, 2007b, 2009). The folk rule can be defined in different ways, for instance, through Kruskal´s algorithm or as the Shapley value of the irreducible game or the optimistic game. In Bergantiños and Gómez-Rúa (2010) we define the rule $F$ in *mcstp* with groups as the Owen value (Owen, 1977) of the irreducible game with groups. Since the Owen value generalizes the Shapley value, $F$ coincides with the folk rule in classical *mcstp*.

Bergantiños and Vidal-Puga (2009) give an axiomatic characterization of the folk rule. In Bergantiños and Gómez-Rúa (2010) we extend it to *mcstp* with groups. Some of the properties are the same and other need to be adapted. We do it by claiming the property twice: once among the groups and the other among the agents inside the same group. Let us clarify this idea with the property of population monotonicity. In classical *mcstp*, population monotonicity says that if a new agent joins the society, no agent of the initial society can be worse off. In *mcstp* with groups population monotonicity is divided in two parts: population monotonicity over groups and population monotonicity over agents. Population monotonicity over groups says that if a new group joins the society, no agent of the initial society can be worse off. Population monotonicity over agents says that if agent $i$ enters in group $G^k$, no agent of group $G^k$ can be worse off. Moreover, if the connection costs between group $G^k$ and the other groups do not change, agents of the other groups must pay the same.

Bergantiños and Vidal-Puga (2007a) give two characterizations of the folk rule. In the first one they prove that the folk rule is the unique rule satisfying cost monotonicity[1], population monotonicity, and equal share of extra costs. Cost monotonicity states that if a network connection cost increases, no agent should pay less. The idea behind equal share of extra costs is the following: consider a problem where the most expensive connection cost for any agent is the cost of connecting to the source. Additionally, the connection cost to the source is the same for all agents. If we assume that this connection cost increases by $x > 0$, then equal share of extra costs states that if agent $i$ pays $f_i$ in the original problem, he must pay $f_i + \frac{x}{n}$ when the cost increases (where $n$ is the number of agents). In the second characterization of the folk rule they replace cost monotonicity by independence of irrelevant trees and population monotonicity by separability. Independence of irrelevant trees states that if two *mcstp* have the same minimum cost spanning tree with the same costs in each arc they belong to, then the rule must coincide in both *mcstp*. Assuming that two subsets of agents, $S$ and $N \backslash S$, can be connected to the source either separately or jointly, separability states that if there are no savings when they are jointly connected to the source, then agents will pay the same in both circumstances.

In this paper we present two characterizations of the rule $F$ in *mcstp* with groups, generalizing the results of Bergantiños and Vidal-Puga (2007a). We first adapt the properties to *mcstp* with groups. Cost monotonicity and independence of irrelevant trees do not need to be adapted (both properties

[1] Cost Monotonicity is named Solidarity in Bergantiños and Vidal-Puga (2007a)

make sense when there are groups). Equal share of extra costs is replaced by coalitional share of extra costs. Under the hypothesis of equal share of extra costs, coalitional share of extra costs states that the groups should share this extra cost $x$ equally among them. Therefore, the extra cost assigned to each group should be shared equally among the agents of this group. Population monotonicity is divided in two properties: population monotonicity over groups and population monotonicity over agents, as studied in Bergantiños and Gómez-Rúa (2010). Finally, separability is also divided in two properties: separability among groups and separability among agents.

The results establish that $F$ is the unique rule in *mcstp* with groups satisfying cost monotonicity, population monotonicity over agents, population monotonicity over groups, and coalitional share of extra costs. We also prove that $F$ is the unique rule in *mcstp* with groups satisfying independence of irrelevant trees, separability among agents, separability among groups, and coalitional share of extra costs; and finally we obtain that the properties used in both results are independent.

The paper is organized as follows. In Section 2 we introduce *mcstp*. In Section 3 we introduce *mcstp* with groups and several properties. In Section 4 we present our results.

# 2 The Problem

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of all possible agents. Given a finite set $N \subset \mathcal{N}$, let $\Pi_N$ be the set of all permutations over $N$. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of elements of $N$ which come before $i$ in the order given by $\pi$, *i.e.*, $Pre\,(i, \pi) = \{j \in N : \pi\,(j) < \pi\,(i)\}$.

We are interested in networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where $N \subset \mathcal{N}$ is finite and 0 is a special node called the *source*. Usually we take $N = \{1, ..., n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ on $N$ represents the cost of direct link between any pair of nodes. We assume that $c_{ij} = c_{ji} \geq 0$ for each $i, j \in N_0$ and $c_{ii} = 0$ for each $i \in N_0$. Since $c_{ij} = c_{ji}$ we work with undirected arcs, *i.e.* $(i, j) = (j, i)$.

We denote the set of all cost matrices over $N$ as $\mathcal{C}^N$. Given $C$, $C' \in \mathcal{C}^N$ we say $C \leq C'$ if $c_{ij} \leq c'_{ij}$ for all $i, j \in N_0$.

A *minimum cost spanning tree problem*, briefly an *mcstp*, is a pair $(N_0, C)$ where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is the cost matrix.

Given an *mcstp* $(N_0, C)$, we define the *mcstp* induced by $C$ in $S \subset N$ as $(S_0, C)$.

A *network* $g$ over $N_0$ is a subset of $\{(i,j) : i,j \in N_0,\ i \neq j\}$. The elements of $g$ are called *arcs*. Given a network $g$ over $N_0$ and $S \subset N_0$ we denote by $g_S$ the network induced by $g$ among the elements of $S$. Namely, $g_S = \{(i,j) \in g : \{i,j\} \subset S\}$.

Given a network $g$ and a pair of nodes $i$ and $j$, a *path* from $i$ to $j$ in $g$ is a sequence of different arcs $\{(i_{h-1}, i_h)\}_{h=1}^{l}$ satisfying $(i_{h-1}, i_h) \in g$ for all $h \in \{1, 2, ..., l\}$, $i = i_0$, and $j = i_l$.

A *tree* is a network such that for all $i \in N$ there is a unique path from $i$ to the source. If $t$ is a tree, we usually write $t = \{(i^0, i)\}_{i \in N}$ where $i^0$ represents the first agent in the unique path in $t$ from $i$ to 0.

Let $\mathcal{G}^N$ denote the set of all networks over $N_0$. Let $\mathcal{G}_0^N$ denote the set of all networks where every agent $i \in N$ is connected to the source, *i.e.* there exists a path from $i$ to 0 in the network.

Given an *mcstp* $(N_0, C)$ and $g \in \mathcal{G}^N$, we define the *cost* associated with $g$ as

$$c(N_0, C, g) = \sum_{(i,j) \in g} c_{ij}.$$

When there is no ambiguity, we write $c(g)$ or $c(C, g)$ instead of $c(N_0, C, g)$.

A *minimum cost spanning tree* for $(N_0, C)$ is a tree $t$ over $N_0$ such that $c(t) = \min_{g \in \mathcal{G}_0^N} c(g)$. It is well-known that a minimum cost spanning tree exists, even though it is not necessarily unique. Given an *mcstp* $(N_0, C)$, we denote the cost associated with any minimum cost spanning tree as $m(N_0, C)$.

Given an *mcstp*, Prim (1957) provides an algorithm for solving the problem of connecting all agents to the source such that the total cost of creating the network is minimal. The idea of this algorithm is simple: starting from the source we construct a network by sequentially adding arcs with the lowest cost and without introducing cycles.

Given a tree $t$, we define the *predecessor set* of a node $i$ in $t$ as $Pre(i,t) = \{j \in N_0 : j \text{ is in the unique path from } i \text{ to the source}\}$. We assume that $i \notin Pre(i,t)$ and $0 \in Pre(i,t)$ when $i \neq 0$. For notational convenience, $Pre(0,t) = \emptyset$. The *distance* from node $i$ to the source in $t$ is the cardinality of $Pre(i,t)$. The *immediate predecessor* of agent $i$ in $t$, denoted as $i^0$, is the node that comes immediately before $i$, that is, $i^0 \in Pre(i,t)$ and $k \in Pre(i,t)$ implies either $k = i^0$ or $k \in Pre(i^0, t)$. Note that $Pre(i^0, t) \subset Pre(i,t)$ and

$Pre(i,t) \backslash Pre(i^0,t) = \{i^0\}$. The *follower set* of an agent $i$ in $t$ is the set $Fol(i,t) = \{j \in N : i \in Pre(j,t)\}$.

Given an *mcstp* $(N_0, C)$ and a minimum cost spanning tree $t$, we define the *minimal problem*[2] $(N_0, C^t)$ associated with $t$ as follows: $c^t_{ij} = \max_{(k,l) \in g_{ij}} \{c_{kl}\}$, where $g_{ij}$ denotes the unique path in $t$ from $i$ to $j$. Even though $g_{ij}$ depends on the choice of $t$, $c^t_{ij}$ is independent of the chosen $t$. Proof of this can be found, for instance, in Aarts and Driessen (1993).

The *irreducible form* of an *mcstp* $(N_0, C)$ is defined as the minimal problem $(N_0, C^*)$ associated with a particular minimum cost spanning tree $t$. If $(N_0, C^*)$ is an *irreducible form*, we say that $C^*$ is an *irreducible matrix*.

A *coalitional game with transferable utility*, briefly a *TU game*, is a pair $(N, v)$ where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. $Sh(N, v)$ denotes the Shapley value (Shapley, 1953) of $(N, v)$.

For each *mcstp* $(N_0, C)$. Bird (1976) introduces the *TU* game $(N, v_C)$. For each coalition $S \subset N$,

$$v_C(S) = m(S_0, C).$$

One of the most important issues addressed in the literature about *mcstp* is how to divide the cost of connecting agents to the source between them. We now briefly introduce some of the rules studied in the literature.

A (*cost allocation*) *rule* is a function $f$ which assigns to each *mcstp* $(N_0, C)$ a vector $f(N_0, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N_0, C) = m(N_0, C)$. As usual, $f_i(N_0, C)$ represents the cost allocated to agent $i$.

Notice that we implicitly assume that the agents build a minimum cost spanning tree. As far as we know, all the rules proposed in the literature make this assumption.

There are several rules studied in the literature. We mention, for instance, the rules studied in Bird (1976), Kar (2002), and Dutta and Kar (2004). In this paper the rule introduced by Feltkamp *et al.* (1994) and called Equal Remaining Obligations rule ($ERO$) will be very important. $ERO$ is called the $P-value$ in Branzei *et al.* (2004).

On the other hand, in Bergantiños and Vidal-Puga (2007a) the rule $\varphi$ is defined as

$$\varphi(N_0, C) = Sh(N, v_{C^*})$$

[2] Bird (1976) denotes it as the *minimal network*.

where $C^*$ is the irreducible matrix associated with $C$. Surprisingly $\varphi$ coincides with $ERO$, a proof can be found, for instance, in Bergantiños and Lorenzo-Freire (2008). This rule is also studied in Bergantiños and Vidal-Puga (2007b, 2009). We call *folk rule* to this rule.

We now present some properties introduced in the literature of the *mcstp*.

*Population Monotonicity:* For all *mcstp* $(N_0, C)$, $S \subset N$, and $i \in S$, we have

$$f_i(N_0, C) \leq f_i(S_0, C).$$

This property implies that if new agents join a "society" no agent from the "initial society" can be worse off.

*Separability:* For all *mcstp* $(N_0, C)$ and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m((N \backslash S)_0, C)$, we have

$$f_i(N_0, C) = \begin{cases} f_i(S_0, C) & \text{if } i \in S \\ f_i\left((N \backslash S)_0, C\right) & \text{if } i \in N \backslash S. \end{cases}$$

Two subsets of agents, $S$ and $N \setminus S$, can be connected to the source either separately or jointly. If there are no savings when they are jointly connected to the source, this property implies that agents will pay the same in both circumstances.

Notice that if a rule $f$ satisfies population monotonicity, then it also satisfies separability.

*Cost Monotonicity:* For all *mcstp* $(N_0, C)$ and $(N_0, C')$ such that $C \leq C'$, we have

$$f(N_0, C) \leq f(N_0, C').$$

This property implies that if a number of connection costs increase and the rest of connection costs (if any) remain the same, no agent can be better off. Notice that cost monotonicity demands agents' contribution to move in the same direction irrespective of their locations on minimum cost spanning trees.

We say that two *mcstp* $(N_0, C)$ and $(N_0, C')$ are *tree-equivalent* if there exists a tree $t$ such that, firstly, $t$ is a minimum cost spanning tree for both $(N_0, C)$ and $(N_0, C')$ and secondly, $c_{ij} = c'_{ij}$ for all $(i, j) \in t$.

*Independence of irrelevant trees:* If two *mcstp* $(N_0, C)$ and $(N_0, C')$ are tree-equivalent,

$$f(N_0, C) = f(N_0, C').$$

Independence of irrelevant trees states that if two *mcstp* have the same minimum cost spanning tree with the same costs in each arc they belong to, then the rule must coincide in both *mcstp*.

Bergantiños and Vidal-Puga (2007a) prove that if a rule $f$ satisfies cost monotonicity, then also satisfies independence of irrelevant trees.

*Equal share of extra costs:* Let $(N_0, C)$ and $(N_0, C')$ be two *mcstp*. Let $c_0, c_0' \geq 0$. Assuming $c_{0i} = c_0$ and $c'_{0i} = c'_0$ for all $i \in N$, $c_0 < c'_0$, and $c_{ij} = c'_{ij} \leq c_0$ for all $i, j \in N$, we have

$$f_i(N_0, C') = f_i(N_0, C) + \frac{c'_0 - c_0}{n}$$

for all $i \in N$.

This property is interpreted as follows: a group of agents $N$ faces a problem $(N_0, C)$ in which all of them have the same connection cost to the source $(c_{i0} = c_0)$ and in which this cost is greater than the connection costs between agents $(c_{ij} \leq c_0)$. Under these circumstances, an optimal network implies that any one agent connects directly to the source, and that the rest connect to the source through this agent. Therefore, they agree that the correct solution is $f(N_0, C)$. Assume that an error was made and that the connection cost to the source is $c'_0 > c_0$. equal share of extra costs states that the agents should share this extra cost $c'_0 - c_0$ equally among them.

Bergantiños and Vidal-Puga (2007a) give the following characterizations of the folk rule $\varphi$.

**Theorem 0:**

$(a)$ $\varphi$ is the unique rule on *mcstp* satisfying independence of irrelevant trees, separability, and equal share of extra costs.

$(b)$ $\varphi$ is the unique rule on *mcstp* satisfying cost monotonicity, population monotonicity, and equal share of extra costs.

In Lemma 0 below we present some results used in the paper. The proof can be found in Bergantiños and Gómez-Rúa (2010) and in Bergantiños and Vidal-Puga (2007a, 2007b, 2009).

**Lemma 0**. $(a)$ Given $(N_0, C, G)$ we can find an minimum cost spanning tree $t$ in $(N_0, C)$ satisfying:

$(i)$ For each $k = 1, ..., m$, $t_{G^k}$ is a minimum cost spanning tree in $(G^k, C)$.

(ii) $t\backslash\left(\bigcup_{k=1}^{m} t_{G^k}\right)$ defined as $\left\{(k,k') : \exists i \in G^k, j \in G^{k'} \text{ with } (i,j) \in t\right\}$ is a minimum cost spanning tree in $\left(G_0, C^G\right)$.

(iii) For each $k = 1, ...., m$ and each $i \in G^k$, $t_{G_k} \cup \{(0,i)\}$ is a minimum cost spanning tree in $(G_0^k, C^\varphi)$.

(b) If $C$ is an irreducible matrix, then for all $S \subset N$ and $i \notin S$ we have that

$$v_C\left(S \cup \{i\}\right) - v_C\left(S\right) = \min_{j \in S_0} \{c_{ij}\}.$$

# 3 Minimum cost spanning tree problems with groups

In Bergantiños and Gómez-Rúa (2010) we introduce the *minimum cost spanning tree problems with groups*. A *mcstp* with groups is a triple $(N_0, C, G)$ where $(N_0, C)$ is a *mcstp*, $G = \{G^1, ..., G^m\}$ is a partition of $N$ and for each $k = 1, ...m$,

$$\max_{i,j \in G^k} \{c_{ij}\} \leq \min_{i \in G^k, j \in N_0 \backslash G^k} \{c_{ij}\}.$$

A *rule* in *mcstp* with groups is a function $f$ assigning to each *mcstp* with groups $(N_0, C, G)$ a vector $f(N_0, C, G) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(N_0, C, G) = m(N_0, C)$.

As in classical *mcstp*, the main objective is to divide the cost associated with a minimum cost spanning tree among the agents in a fair way.

In Bergantiños and Gómez-Rúa (2010) we introduce a rule, $F$ as the unique rule in *mcstp* with groups satisfying a set of properties. The rule is defined as follows: we first give the intuitive idea. This rule can be considered as a two step rule. In the first step, we compute the amount that each group should pay in order to be connected to the source. We do this by applying the rule $\varphi$ defined in Bergantiños and Vidal-Puga (2007a).

In the second step we decide the amount that each agent of each group has to pay. For each group $G^k$, we consider the *mcstp* inside each group $\left(G_0^k, C^\varphi\right)$. In this *mcstp*, the connection cost between two agents of $G^k$ is the same as in $C$ but the connection cost between any agent of $G^k$ and the source is the amount computed for $G^k$ in the first step.

We now present the definition formally. Given the *mcstp* with groups $(N_0, C, G)$ we define the *mcstp* among groups $\left(G_0, C^G\right)$ as follows:

- $G_0 = \{G^0, G^1, ..., G^m\}$ where $G^0 = 0$. In order to simplify the notation, we often use $k$ instead of $G^k$.
- $C^G$ is the cost matrix and for each $G^k, G^{k'} \in G_0$ the connection cost between $G^k$ and $G^{k'}$ is denoted by

$$c^G_{kk'} = \min_{i \in G^k, j \in G^{k'}} \{c_{ij}\}.$$

Let $(N_0, C, G)$ *be a mcstp* with groups. For each $i \in G^k$,

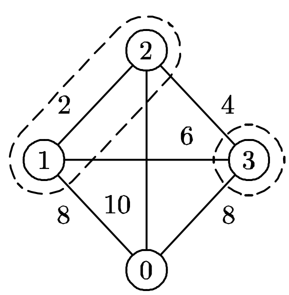
$$F_i(N_0, C, G) = \varphi_i\left(G_0^k, C^\varphi\right)$$

where

$$c^\varphi_{jj'} = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ \varphi_k\left(G_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$$

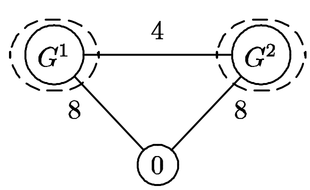
In order to clarify the idea, consider the following example:

**Example 1:** Let the *mcstp* $(N_0, C, G)$ with groups where $N = \{1, 2, 3\}$, $G = \{G^1, G^2\}$, $G^1 = \{1, 2\}$, $G^2 = \{3\}$, and matrix $C$ which is represented in the following figure:




In this case, $m(N_0, C) = 14$.

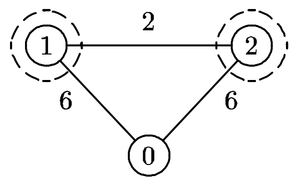
In the first step we define the *mcstp* among groups, then, we have the *mcstp* $(G_0, C^G)$, where $G = \{G^1, G^2\}$ and the cost matrix $C^G$ is represented in the following figure:



Note that $m(G_0, C^G) = 12$. Applying $\varphi$ to this problem, we obtain that $\varphi_1(G_0, C^G) = \varphi_2(G_0, C^G) = 6$.

In the second step we distribute the amount assigned to each group among all its members.

As $G^2 = \{3\}$, we can already conclude that $F_3(N_0, C, G) = 6$. Now we distribute $\varphi_1(G_0, C^G) = 6$ among the agents 1 and 2. Let define the problem $(G_0^1, C^\varphi)$. In this case, the cost matrix $C^\varphi$ is represented in the following figure:




Applying $\varphi$ to this problem, we conclude that $F_1(N_0, C, G) = \varphi_1(G_0^1, C^\varphi) = 4$ and $F_2(N_0, C, G) = \varphi_2(G_0^1, C^\varphi) = 4$. Then, $F(N_0, C, G) = (4, 4, 6)$.

In this paper we follow the axiomatic approach and we present two axiomatic characterizations for the rule $F$. Our idea is to generalize the axiomatic characterizations of the folk rule $\varphi$ given by Bergantiños and Vidal-Puga (2007a) and presented in Theorem 0.

In order to obtain these axiomatic approaches, we now adapt these properties to *mcstp* with groups.

Cost monotonicity and independence of irrelevant trees can be formulated in a similar way. Population monotonicity, separability, and equal share of extra costs should be adapted. The main idea for adapting population monotonicity and separability is claiming both twice, once among the groups and other among agents inside the same group. Equal share of extra costs is adapted in a different way explained below.

Bergantiños and Gómez-Rúa (2010) introduce the following properties by adapting population monotonicity.

*Population monotonicity over groups:* For all *mcstp* with groups $(N_0, C, G)$, all $G^k \in G$, and all $i \in N \backslash G^k$,

$$f_i (N_0, C, G) \leq f_i \left( \left( N \backslash G^k \right)_0, C, G \backslash G^k \right).$$

The population monotonicity over agents will show what happens when an agent enters in a group. We claim that no agent of the initial group can be worse off.

Assume that after the entrance of agent $i$ in group $G^k$ the minimum connection cost between group $G^k$ and the rest of the groups did not change, *i.e.*, for each $G^l \in G_0 \backslash \left\{G^k\right\}$, $\min\limits_{j \in G^k, j' \in G^l} \{c_{jj'}\} = \min\limits_{j \in G^k \backslash \{i\}, j' \in G^l} \{c_{jj'}\}$. Since we are assuming that the amount paid by a group should not depend on the internal characteristics of the other groups, and the entrance of agent $i$ does not change the connection cost among groups, we claim that agents of the others groups must pay the same.

*Population monotonicity over agents:* For all *mcstp* with groups $(N_0, C, G)$, all $G^k \in G$, and all $i \in G^k$ such that $G^k \backslash \{i\} \neq \varnothing$,

$$f_j(N_0, C, G) \leq f_j\left((N \backslash \{i\})_0, C, G^{-i}\right) \text{ if } j \in G^k \backslash \{i\}.$$

Additionally, if for each $G^l \in G \backslash \left\{G^k\right\}$, $\min\limits_{j \in G^k, j' \in G^l} \{c_{jj'}\} = \min\limits_{j \in G^k \backslash \{i\}, j' \in G^l} \{c_{jj'}\}$, then

$$f_j(N_0, C, G) = f_j\left((N \backslash \{i\})_0, C, G^{-i}\right) \text{ if } j \in N \backslash G^k.$$

We now define two properties of separability.

*Separability among groups*: For all *mcstp* $(N_0, C, G)$ and $R \subset G$ satisfying $m(N_0, C) = m\left(\left(\cup_{k \in R} G^k\right)_0, C\right) + m\left(\left(\cup_{k \notin R} G^k\right)_0, C\right)$, we have

$$f_i(N_0, C, G) = \begin{cases} f_i\left(\left(\cup_{k \in R} G^k\right)_0, C, \left\{\left\{G^k\right\}\right\}_{k \in R}\right) & \text{if } i \in \cup_{k \in R} G^k \\ f_i\left(\left(\cup_{k \notin R} G^k\right)_0, C, \left\{\left\{G^k\right\}\right\}_{k \notin R}\right) & \text{if } i \in N \backslash \cup_{k \in R} G^k. \end{cases}$$

This property ensures the following: two subsets of groups $R$ and $G \backslash R$ can be connected to the source either separately or jointly. If there is no savings when they are jointly connected to the source, this property implies that agents in each group will pay the same in both circumstances.

*Separability among agents*: For all *mcstp* $(N_0, C, \{N\})$ and $S \subset N$ satisfying $m(N_0, C) = m(S_0, C) + m\left((N \setminus S)_0, C\right)$, we have

$$f_i(N_0, C, \{N\}) = \begin{cases} f_i(S_0, C, \{S\}) & \text{if } i \in S \\ f_i\left((N \backslash S)_0, C, \{N \backslash S\}\right) & \text{if } i \in N \backslash S. \end{cases}$$

Assume that there is a unique group. Two subsets of agents $S$ and $N \backslash S$ can be connected to the source either separately or jointly. If there is no savings when they are jointly connected to the source, this property implies that agents will pay the same in both circumstances.

Finally, we adapt equal share of extra costs to *mcstp* with groups.

*Coalitional share of extra costs*: Let $(N_0, C, G)$ and $(N_0, C', G)$ be two *mcstp* with groups. Let $c_0, c_0' \geq 0$. Assume that $c_{0i} = c_0$ and $c_{0i}' = c_0'$ for all $i \in N$, $c_0 < c_0'$, and $c_{ij} = c_{ij}' \leq c_0$ for all $i, j \in N$. For each $i \in G^k \in G$

$$f_i (N_0, C', G) = f_i (N_0, C, G) + \frac{c_0' - c_0}{m |G^k|}$$

Coalitional share of extra costs states that when an extra cost appears, $c_0' - c_0$, the groups should share this extra cost equally among them. Besides, the extra cost assigned to each group should be shared equally among the agents of this group.

# 4 Characterization

In this section we generalize the results presented in Theorem 0 (Bergantiños and Vidal-Puga, 2007a) to *mcstp* with groups, using the properties presented in the previous section. We obtain two characterization results for the rule $F$ introduced in Bergantiños and Gómez-Rúa (2010).

**Theorem 1**. $(a)$ $F$ is the unique rule satisfying independence of irrelevant trees, separability among groups, separability among agents, and coalitional share of extra costs.

$(b)$ $F$ is the unique rule satisfying cost monotonicity, population monotonicity over groups, population monotonicity over agents, and coalitional share of extra costs.

**Proof of Theorem 1**. We now briefly discuss the relationship between our proof and the proof of Theorems 4.1 and 4.2 in Bergantiños and Vidal-Puga (2007a). The existence part of both proofs is qualitative different. The proof of 4.1 in Bergantiños and Vidal-Puga (2007a) is made by doing some algebra. Our proof is made by using the properties of the folk rule. The uniqueness part is qualitative similar but technically different. Both proofs are qualitative similar because they use induction on the number of agents. When there is only one agent the result holds trivially. When there are $n$

agents, using the list of properties, we relate for each agent the allocation proposed by the rule in the problem with $n$ agents, with the allocation proposed by the rule in a problem with less agents. Both proofs are technically different. Let me give you two examples. In this proof we consider a tree $t$ as in Lemma 0 $(a)$, whereas in Bergantiños and Vidal-Puga (2007a) a different tree is used. When there is a unique agent who connects directly to the source in $t$ we should consider three cases, whereas Bergantiños and Vidal-Puga (2007a) consider a unique case, which is easier.

We now prove that $F$ satisfies all the properties mentioned above. Bergantiños and Gómez-Rúa (2010) prove that $F$ satisfies population monotonicity over groups and population monotonicity over agents. The other properties are a consequence of the following claims:

**Claim 1**. $F$ satisfies cost monotonicity.

**Proof of Claim 1**. Let $(N_0, C, G)$ and $(N_0, C', G)$ such that $C \leq C'$. Consider the problems among coalitions, $(G_0, C^G)$ and $(G_0, C'^G)$. By definition of $C^G$ and $C'^G$, we have that $c^G_{kk'} \leq c'^G_{kk'}$ for all $G^k, G^{k'} \in G_0$. Then $C^G \leq C'^G$. Since $\varphi$ satisfies cost monotonicity (Theorem 0), $\varphi_k(G_0, C^G) \leq \varphi_k(G_0, C'^G)$ for all $G^k \in G_0$.

Consider now the problem inside a coalition $(G^k_0, C^\varphi)$, $k = 1, ..., m$. In this case, $c^\varphi_{ij} = c_{ij} \leq c'_{ij} = c'^\varphi_{ij}$ for all $i, j \in G^k$ and $c^\varphi_{0i} = \varphi_k(G_0, C^G) \leq \varphi_k(G_0, C'^G) = c'^\varphi_{0i}$ for all $i \in G^k$. Then, $C^\varphi \leq C'^\varphi$, since $\varphi$ satisfies cost monotonicity (Theorem 0), we have

$$F_i(N_0, C, G) = \varphi_i(G^k_0, C^\varphi) \leq \varphi_i(G^k_0, C'^\varphi) = F_i(N_0, C', G)$$

for all $i \in G^k$.■

**Claim 2**. Let $f$ be a rule satisfying population monotonicity over groups, then $f$ also satisfies separability among groups.

**Proof of Claim 2**. Bergantiños and Vidal-Puga (2007a) prove that if a rule $f$ satisfies population monotonicity, then $f$ also satisfies separability. The proof of this claim is similar and we omit it.■

**Claim 3**. Let $f$ be a rule satisfying population monotonicity over agents, then $f$ also satisfies separability among agents.

**Proof of Claim 3**. It is similar to the proof of Claim 2 and we omit it.■

**Claim 4**. $F$ satisfies coalitional share of extra costs.

**Proof of Claim 4**. Let $(N_0, C, G)$ and $(N_0, C', G)$ be under the conditions of coalitional share of extra costs. Consider the problems among groups $(G_0, C^G)$ and $(G_0, C'^G)$. Thus, $c^G_{0k} = \min_{j \in G^k}\{c_{0j}\} = c_0$ and $c'^G_{0k} = c'_0$ for all $G^k \in G$. Moreover, $c^G_{kk'} = \min_{j \in G^k, j' \in G^{k'}}\{c_{jj'}\} \leq c_0$, $c'^G_{kk'} = \min_{j \in G^k, j' \in G^{k'}}\{c'_{jj'}\} \leq c'_0$, and $c^G_{kk'} \leq c'^G_{kk'}$ for all $G^k, G^{k'} \in G$. Reasoning in the same way, we obtain that $c'^G_{0k} = c'_0$ and $c'^G_{kk'} = c^G_{kk'}$. Thus, the problems $(G_0, C^G)$ and $(G_0, C'^G)$ are under the conditions of equal share of extra costs. Since $\varphi$ satisfies equal share of extra costs (Theorem 0), we have that

$$\varphi_k(G_0, C'^G) = \varphi_k(G_0, C^G) + \frac{c'_0 - c_0}{m} \tag{1}$$

for all $G^k \in G$.

Consider now the problems inside each group $G^k \in G$, $(G^k_0, C^\varphi)$ and $(G^k_0, C'^\varphi)$. By definition, $c^\varphi_{0i} = \varphi_k(G_0, C^G) := c^\varphi_0$ for all $i \in G^k$ and $c^\varphi_{ij} = c_{ij}$ for all $i, j \in G^k$. Additionally, in the proof of Lemma 0 (a) (iii) (Bergantiños and Gómez-Rúa, 2010) we prove that $c^\varphi_0 \geq \max_{j,j' \in G^k}\{c^\varphi_{jj'}\}$. By reasoning in the same way we obtain that $c'^\varphi_{0i} = \varphi_k(G_0, C'^G) = c'^\varphi_0$ for all $i \in G^k$, $c'^\varphi_{ij} = c'_{ij} = c_{ij}$ for all $i, j \in G^k$ and $c'^\varphi_0 \geq \max_{j,j' \in G^k}\{c'^\varphi_{jj'}\}$. Likewise, if we take expression (1) into account , we have that $c^\varphi_0 < c'^\varphi_0$, and $c'^\varphi_0 - c^\varphi_0 = \varphi_k(G_0, C'^G) - \varphi_k(G_0, C^G) = \frac{c'_0 - c_0}{m}$. Thus, $(G^k_0, C^\varphi)$ and $(G^k_0, C'^\varphi)$ are also under the conditions of equal share of extra costs. Since $\varphi$ satisfies equal share of extra costs (Theorem 0), we obtain that

$$\begin{aligned} F_i(N_0, C', G) &= \varphi_i(G^k_0, C'^\varphi) = \varphi_i(G^k_0, C^\varphi) + \frac{c'^\varphi_0 - c^\varphi_0}{|G^k|} \\ &= \varphi_i(G^k_0, C^\varphi) + \frac{c'_0 - c_0}{m\,|G^k|} \\ &= F_i(N_0, C, G) + \frac{c'_0 - c_0}{m\,|G^k|} \end{aligned}$$

for all $i \in G^k$. ■

We now prove the uniqueness in each of the two parts.

$(a)$ We apply an induction argument over the number of agents $|N|$. If $|N| = 1$ the result is trivial. Assume that the result holds for less than $n$ agents. We prove it for $n$ agents.

Let $f$ be a rule satisfying these properties. Given a *mcstp* with groups $(N_0, C, G)$ we prove that $f(N_0, C, G) = F(N_0, C, G)$.

Let $t = \{(i^0, i)\}_{i \in N}$ be a minimum cost spanning tree in $(N_0, C)$ as in Lemma 0 (a).

Assume that there is at least two agents who connect directly to the source in $t$. Namely, $|\{i \in N : (0, i)\} \in t| \geq 2$. Let $j, j'$ be such that $(0, j) \in t$ and $(0, j') \in t$,

We define $S = Fol(j, t) \cup \{j\}$. Thus, $m(N_0, C) = m(S_0, C) + m((N \backslash S)_0, C)$. Since $t$ satisfies the conditions of Lemma 0 (a), there exists $R \subset G$ such that $S = \cup_{k \in R} G^k$. Since $f$ and $F$ satisfy separability among groups, it is not difficult to prove that $f_i(N_0, C, G) = F_i(N_0, C, G)$ for all $i \in N$.

Assume now that there is a unique agent who connect directly to the source in $t$. We define $x = \max\{c_{i^0 i} : (i^0, i) \in t\}$ and $x' = \max\{c_{i^0 i} : (i^0, i) \in t \text{ and } c_{i^0 i} < x\}$.

We consider several cases.

1. There exists $(j^0, j) \in t$ such that $j \in G^k$, $j^0 \in G^{k'}$, $k \neq k'$, and $c_{j^0 j} = x$.

   By the definition of $C^*$ as the minimal problem associated with $t$, we can deduce that $t$ is also a minimum cost spanning tree in $C^*$. Since $(j^0, j)$ is in the unique path in $t$ from $j$ to 0, we deduce that $c^*_{0j} = x$.

   Since $f$ and $F$ satisfy independence of irrelevant trees, $f(N_0, C, G) = f(N_0, C^*, G)$ and $F(N_0, C, G) = F(N_0, C^*, G)$. So we can focus in the problem $(N_0, C^*, G)$.

   Let $t^* = t \backslash \{(j^0, j)\} \cup \{(0, j)\}$. It is clear that $c(C^*, t^*) = c(C^*, t)$.

   Then, $t^*$ is also a minimum cost spanning tree in $C^*$. Besides, there are at least two agents connected directly to the source in $t^*$. Thus, $f(N_0, C^*, G) = F(N_0, C^*, G)$.

2. We are not in Case 1 and there exists $(j^0, j) \in t$ such that $j, j^0 \in G^k$ and $c_{j^0 j} = x$.

   We first prove that $G = \{N\}$. Suppose not. By definition of $C$, there exists an arc $(i^0, i) \in t$ with $i \in G^k$, $i^0 \in N_0 \backslash G^k$ such that $c_{i^0 i} \geq c_{j^0 j}$. Since $c_{j^0 j} = x$, $c_{i^0 i} = c_{j^0 j} = x$. So we are in Case 1, which is a contradiction.

   We consider $C^*$. Using arguments similar to those used in Case 1, separability among agents instead separability among groups, we can deduce that $f(N_0, C, G) = F(N_0, C, G)$.

3. There exists $j \in N$ such that $(0, j) \in t$ and $c_{0j} = x > c_{i^0 i}$ for all $(i^0, i) \in t$ with $\{i^0, i\} \subset N$. Notice that we are not in any of the previous cases.

   We can compute $C^*$ as the minimal problem associated to $t$. Since $(0, j)$ is the unique arc in $t$ containing the source, for each $i \in N$, $(0, j)$ is in the unique path in $t$ from 0 to $i$. Thus, $c^*_{0i} = c_{0j} = x$. Additionally, for each $\{i, i'\} \subset N$, $(0, j)$ is not in the unique path in $t$ from $i$ to $i'$. Thus, $c^*_{ii'} \leq x' < x$.

   Since $f$ and $F$ satisfy independence of irrelevant trees, $f(N_0, C, G) = f(N_0, C^*, G)$ and $F(N_0, C, G) = F(N_0, C^*, G)$.

   Let $C'$ be such that $c'_{0i} = x'$ for all $i \in N$ and $c'_{ih} = c^*_{ih}$ otherwise. Applying Prim's algorithm is easy to see that $t$ is also a minimum cost spanning tree in $C'$. Furthermore, $C^*$ and $C'$ are under the hypothesis of coalitional share of extra costs. Since $f$ and $F$ satisfy coalitional share of extra costs, it is enough to prove that $f(N_0, C', G) = F(N_0, C', G)$.

   Let $(k^0, k) \in t$ be such that $c_{k^0 k} = x'$. Thus, $c^*_{k^0 k} = c'_{k^0 k} = x'$. Let $t' = t \backslash \{(k^0, k)\} \cup \{(0, k)\}$. It is clear that $c(C', t') = c(C', t)$. Then, $t'$ is also a minimum cost spanning tree in $C'$.

   Two cases are possible. First, we can take $k^0 \in G^l$ and $k \in N \backslash G^l$. Then, $t'$ satisfies the conditions of Lemma 0 (a). Moreover, there are at least two agents connected directly to the source in $t'$. Thus, $f(N_0, C', G) = F(N_0, C', G)$. Second, for any arc $(k^0, k) \in t$ with $c_{k^0 k} = x'$ we have that $k, k^0 \in G^l$. Using arguments similar to those used in Case 2 we can deduce that $G = \{N\}$ and $f(N_0, C', G) = F(N_0, C', G)$.■

($b$) Since cost monotonicity implies independence of irrelevant trees, population monotonicity over groups implies separability among groups, and population monotonicity over agents implies separability among agents, it is a trivial consequence of part ($a$).■

**Proposition 1**. The properties used in Theorem 1 (a) and (b) are independent.

**Proof:** We proof that if we remove some of the properties of Theorem 1 (a) and (b), we can find more rules satisfying the other properties.

There exist rules satisfying cost monotonicity (and hence independence of irrelevant trees), population monotonicity over groups (and hence separability among groups), population monotonicity over agents (and hence separability among agents), and failing coalitional share of extra costs. For instance the rule $f^1$ defined as follows. Given $T \subset \mathcal{N}$, let $\pi^N$ denote the order induced in $N$ by the index of the agents. Namely, given $i, j \in N$, $\pi^N(i) < \pi^N(j)$ if and only if $i < j$. For each *mcstp* $(N_0, C)$ and $i \in N$ we define

$$\psi_i(N_0, C) = v_{C^*}\left(Pre\left(i, \pi^N\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi^N\right)\right).$$

Let $(N_0, C, G)$ be a *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^1(N_0, C, G) = \psi_i\left(G_0^k, C^\varphi\right).$$

There exist rules satisfying cost monotonicity (and hence independence of irrelevant trees), population monotonicity over groups (and hence separability among groups), coalitional share of extra costs and failing separability among agents (and hence population monotonicity over agents). For instance, the rule $f^2$ defined as follows. For each *mcstp* $(N_0, C)$ and $i \in N$ we define the equal division rule as

$$E_i(N_0, C) = \frac{m(N_0, C)}{|N|}.$$

Let $(N_0, C, G)$ be an *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^2(N_0, C, G) = E_i\left(G_0^k, C^\varphi\right).$$

There exist rules satisfying cost monotonicity (and hence independence of irrelevant trees), population monotonicity over agents (and hence separability among agents), coalitional share of extra costs and failing separability among groups (and hence population monotonicity over groups). For instance, the rule $f^3$ defined as follows. Let $(N_0, C, G)$ be a *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^3(N_0, C, G) = \varphi_i\left(G_0^k, C^E\right)$$

where $c_{jj'}^E = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ E_k\left(G_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$.

There exist rules satisfying population monotonicity over groups (and hence separability among groups), population monotonicity over agents (and hence separability among agents), coalitional share of extra costs and failing cost monotonicity (and hence independence of irrelevant trees). For instance, the rule $f^4$ defined as follows. Let $(N_0, C, G)$ be a *mcstp* with groups. We say that $\pi \in \Pi_N$ is *admissible* if the following conditions are satisfied: i) given $i \in G^k$, $j \in G^{k'}$, $k \neq k'$, if $\pi(i) < \pi(j)$, then $c^G_{0k} \leq c^G_{0k'}$; ii) given $i, j \in G^k$, if $\pi(i) < \pi(j)$, then $c_{0i} \leq c_{0j}$ and iii) if $\pi(i) < \pi(j) < \pi(l)$, with $i, l \in G^k$, then, $j \in G^k$. We denote by $\Pi'^N$ the set of all permutations over $N$ that are admissible. In an intuitive way: we first order the groups by decreasing connection cost to the source; and then the agents inside groups following the same idea.

Thus, for all $i \in N$,

$$f^4_i(N_0, C, G) = \frac{1}{|\Pi'_N|} \sum_{\pi \in \Pi'_N} v_{C^*}(Pre(i,\pi) \cup \{i\}) - v_{C^*}(Pre(i,\pi)].$$

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# 7 Appendix (not for publication)

We prove Proposition 1, which is a consequence of the following claims.

**Claim 1** There exist rules satisfying cost monotonicity (and hence independence of irrelevant trees), population monotonicity over groups (and hence separability among groups), population monotonicity over agents (and hence separability among agents), and failing coalitional share of extra costs: Let define the rule $f^1$ as follows. Given $T \subset \mathcal{N}$, let $\pi^N$ denote the order induced in $N$ by the index of the agents. Namely, given $i, j \in N$, $\pi^N(i) < \pi^N(j)$ if and only if $i < j$. For each *mcstp* $(N_0, C)$ and $i \in N$ we define

$$\psi_i(N_0, C) = v_{C^*}\left(Pre\left(i, \pi^N\right) \cup \{i\}\right) - v_{C^*}\left(Pre\left(i, \pi^N\right)\right).$$

Let $(N_0, C, G)$ be an *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^1(N_0, C, G) = \psi_i\left(G_0^k, C^\varphi\right).$$

**Claim 2** There exist rules satisfying cost monotonicity (and hence independence of irrelevant trees), population monotonicity over groups (and hence separability among groups), coalitional share of extra costs and failing separability among agents (and hence population monotonicity over agents): Let define the rule $f^2$ as follows. For each *mcstp* $(N_0, C)$ and $i \in N$ we define the equal division rule as

$$E_i(N_0, C) = \frac{m(N_0, C)}{|N|}.$$

Let $(N_0, C, G)$ be a *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^2(N_0, C, G) = E_i\left(G_0^k, C^\varphi\right).$$

**Claim 3** There exist rules satisfying cost monotonicity (and hence independence of irrelevant trees), population monotonicity over agents (and hence separability among agents), coalitional share of extra costs and failing separability among groups (and hence population monotonicity over groups).

We define the rule $f^3$ as follows. Let $(N_0, C, G)$ be a *mcstp* with groups and $i \in G^k$. Thus,

$$f_i^3(N_0, C, G) = \varphi_i\left(G_0^k, C^E\right)$$

where $c_{jj'}^E = \begin{cases} c_{jj'} & \text{if } 0 \notin \{j, j'\} \\ E_k\left(G_0, C^G\right) & \text{if } 0 \in \{j, j'\}. \end{cases}$.

**Claim 4** There exist rules satisfying population monotonicity over groups (and hence separability among groups), population monotonicity over agents (and hence separability among agents), coalitional share of extra costs and failing cost monotonicity (and hence independence of irrelevant trees): We define the rule $f^4$ as follows. Let $(N_0, C, G)$ be a *mcstp* with groups. We say that $\pi \in \Pi_N$ is *admissible* if the following conditions are satisfied: i) given $i \in G^k$, $j \in G^{k'}$, $k \neq k'$, if $\pi(i) < \pi(j)$, then $c_{0k}^G \leq c_{0k'}^G$; ii) given $i, j \in G^k$, if $\pi(i) < \pi(j)$, then $c_{0i} \leq c_{0j}$ and iii) if $\pi(i) < \pi(j) < \pi(l)$, with $i, l \in G^k$, then, $j \in G^k$. We denote by $\Pi'^N$ the set of all permutations over $N$ that are admissible. In an intuitive way: we first order the groups by decreasing connection cost to the source; and then the agents inside groups following the same idea.

Thus, for all $i \in N$,

$$f_i^4(N_0, C, G) = \frac{1}{|\Pi'_N|} \sum_{\pi \in \Pi'_N} v_{C^*}(Pre(i, \pi) \cup \{i\}) - v_{C^*}(Pre(i, \pi)].$$

**Proof of Claim 1:**

1. $f^1$ satisfies cost monotonicity. Let $(N_0, C, G)$ and $(N_0, C', G)$ be such that $C \leq C'$. Consider the problems among coalitions, $(G_0, C^G)$ and $(G_0, C'^G)$. Then, $C^G \leq C'^G$. Since $\varphi$ satisfies cost monotonicity (Theorem 0), $\varphi_k(G_0, C^G) \leq \varphi_k(G_0, C'^G)$ for all $G^k \in G_0$.

Consider now the problem inside a coalition $G_0^k$ with $k = 1, ..., m$. Then, $C^\varphi \leq C'^\varphi$. By definition of the irreducible matrix, $(C^\varphi)^* \leq (C'^\varphi)^*$. By Lemma 0 (b), for all $i \in G^k$,

$$\psi_i\left(G_0^k, C^\varphi\right) \leq \psi_i\left(G_0^k, C'^\varphi\right).$$

2. $f^1$ satisfies population monotonicity over groups. See Bergantiños and Gómez-Rúa (2010).

3. $f^1$ satisfies population monotonicity over agents. See Bergantiños and Gómez-Rúa (2010).

4. $f^1$ fails coalitional share of extra costs. Consider the *mcstp* with groups where $N = \{1,2\}$, $G = \{\{i\}\}_{i \in N}$, and matrices

$$C = \begin{pmatrix} 0 & 10 & 10 \\ 10 & 0 & 2 \\ 10 & 2 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 12 & 12 \\ 12 & 0 & 2 \\ 12 & 2 & 0 \end{pmatrix}.$$

In this case, $f^1(N_0, C, G) = (10, 2)$ and $f^1(N_0, C', G) = (12, 2)$.■

**Proof of Claim 2:**

1. $f^2$ satisfies cost monotonicity. Let $(N_0, C, G)$ and $(N_0, C', G)$ be such that $C \leq C'$ Then $C^G \leq C'^G$. Since $\varphi$ satisfies cost monotonicity (Theorem 0), $\varphi_k(G_0, C^G) \leq \varphi_k(G_0, C'^G)$ for all $G^k \in G_0$.

Consider now the problem inside a coalition $(G_0^k, C^\varphi)$ with $k = 1, ..., m$. Then, $C^\varphi \leq C'^\varphi$ and hence

$$f_i^2(N_0, C, G) = E_i(G_0^k, C^\varphi) \leq E_i(G_0^k, C'^\varphi) = f_i^2(N_0, C', G)$$

for all $i \in G^k$.

2. $f^2$ satisfies population monotonicity over groups. Let $G^k \in G$. Since $\varphi$ satisfies population monotonicity (Theorem 0), $\varphi_l(G_0, C^G) \leq \varphi_l((G \backslash G^k)_0, C^{G \backslash G^k})$ for all $l \neq k$. Furthermore, $((G \backslash G^k)_0, C^G) = ((G \backslash G^k)_0, C^{G \backslash G^k})$

Let $C'^\varphi$ denote the matrix $C^\varphi$ associated with the problem $\left(\left(N \backslash G^k\right)_0, C, G \backslash G^k\right)$.

Let $G^l \in G \backslash G^k$. For all $i, j \in G^l$, $c_{ij}^\varphi = c_{ij}'^\varphi$. For all $i \in G^l$,

$$c_{0i}^\varphi = \varphi_l(G_0, C^G) \leq \varphi_l((G \backslash G^k)_0, C^{G \backslash G^k}) = c_{0i}'^\varphi.$$

That is $C^\varphi \leq C'^\varphi$. Then, $E_i(G_0^l, C^\varphi) \leq E_i(G_0^l, C'^\varphi)$ and hence,

$$f_i^2(N_0, C, G) \leq f_i^2(\left(N \backslash G^k\right)_0, C, G \backslash G^k).$$

3. $f^2$ satisfies coalitional share of extra costs. The proof is similar than the one for $F$ and we omit it.

4. $f^2$ fails separability among agents. Consider the *mcstp* with groups where $N = \{1,2,3\}$, $G = \{N\}$, and matrix

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

In this case, it is clear that $m(N_0, C) = m(\{1,2\}_0, C) + m(\{3\}_0, C)$. However, $f^2(N_0, C, G) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $f^2(\{1,2\}_0, C) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $f^2(\{3\}_0, C) = 1$.■

**Proof of Claim 3**

1. $f^3$ satisfies cost monotonicity. Let $(N_0, C, G)$ and $(N_0, C', G)$ such that $C \leq C'$. Then $C^G \leq C'^G$. Hence, $E_k(G_0, C^G) \leq E_k(G_0, C'^G)$ for all $G^k \in G_0$.

Consider now the problem inside a coalition $(G_0^k, C^E)$ with $k = 1, ..., m$. Then, $C^E \leq C'^E$. Since $\varphi$ satisfies cost monotonicity (Theorem 0),

$$f_i^3 (N_0, C, G) = \varphi_i(G_0^k, C^E) \leq \varphi_i(G_0^k, C'^E) = f_i^3 (N_0, C', G)$$

for all $i \in G^k$.

2. $f^3$ satisfies population monotonicity over agents. Let $G^k \in G$ and $i \in G^k$. By convenience, let us denote as $C'$ the cost matrix $C$ restricted to the problem $((N \backslash \{i\})_0, C, G^{-i})$. Notice that $C'$ coincides with $C$ on the agents of $(N \backslash \{i\})_0$.

We consider several cases:

a) Assume that $c_{kl}^G = c_{kl}'^G$ for all $l \in \{0, 1, ..., m\}$. Thus, $E_l\left(G_0, C^G\right) = E_l\left(G^{-i}, C'^{G^{-i}}\right)$ for all $l = 1, ..., m$.

- Let $j \in G^k \backslash \{i\}$. Since $\varphi$ satisfies population monotonicity (Theorem 0), $\varphi_j\left(G_0^k, C^E\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C^E\right)$. Then,

$$\begin{aligned} f_j^3(N_0, C, G) &= \varphi_j\left(G_0^k, C^E\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C^E\right) \\ &= \varphi_j\left((G^k \backslash \{i\})_0, C'^E\right) \\ &= f_j^3((N \backslash \{i\})_0, C', G^{-i}). \end{aligned}$$

- Let $G^l \in G$ such that $l \neq k$. Then, $c_{jj'}^E = c_{jj'}'^E$ for all $j, j' \in G^l \cup \{0\}$. Hence,

$$f_j^3(N_0, C, G) = f_j^3((N \backslash \{i\})_0, C', G^{-i}).$$

b) Assume that $c_{kk'}^G \neq c_{kk'}'^G$ for some $k' \in \{0, 1, ..., m\}$. Then, $c_{kk'}^G < c_{kk'}'^G$. Furthermore, $c_{ll'}^G \leq c_{ll'}'^G$ for all $l, l' \in \{0, 1, ..., m\}$.

Then, $E_k\left(G_0, C^G\right) \leq E_k\left(G_0, C'^G\right)$, $c_{jj'}^E = c_{jj'}'^E$ for all $j, j' \in G^k \backslash \{i\}$, and $c_{0j}^E \leq c_{0j}'^E$ for all $j \in G^k \backslash \{i\}$. Since $\varphi$ satisfies cost monotonicity (Theorem 0), $\varphi_j\left((G^k \backslash \{i\})_0, C^E\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C'^E\right)$ for all $j \in G^k \backslash \{i\}$.

Since $\varphi$ satisfies population monotonicity (Theorem 0), $\varphi_j\left(G_0^k, C^E\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C^E\right)$. Then, for all $j \in G^k \backslash \{i\}$

$$\begin{aligned} f_j^3(N_0, C, G) &= \varphi_j\left(G_0^k, C^E\right) \leq \varphi_j\left((G^k \backslash \{i\})_0, C^E\right) \\ &\leq \varphi_j\left((G^k \backslash \{i\})_0, C'^E\right) \\ &= f_j^3((N \backslash \{i\})_0, C', G^{-i}). \end{aligned}$$

3. $f^3$ satisfies coalitional share of extra costs. The proof is similar than the one for $F$ and we omit it.

4. $f^3$ fails separability among groups, consider the *mcstp* with groups where $N = \{1, 2, 3\}$, $G = \{\{i\}\}_{i \in N}$, and matrix

$$C = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

In this case, it is clear that $m(N_0, C) = m(\{1, 2\}_0, C) + m(\{3\}_0, C)$. However, $f^3(N_0, C, G) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $f^3(\{1, 2\}_0, C) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $f^3(\{3\}_0, C) = 1$.■

**Proof of Claim 4:**

1. $f^4$ satisfies population monotonicity over groups. We can prove this by performing some computations. Even the proof is not trivial, we do not believe it is especially relevant and we omit it.

2. $f^4$ satisfies population monotonicity over agents. We can prove this by performing some computations. Even the proof is not trivial, we do not believe it is especially relevant and we omit it.

3. $f^4$ satisfies coalitional share of extra costs. Let $(N_0, C, G)$ and $(N_0, C', G)$ be two *mcstp* with groups under the conditions of coalitional share of extra costs. Then, a permutation $\pi \in \Pi'^N$ is admissible if given $i, i' \in G^k \in G$ and $j \in N$ with $\pi(i) < \pi(j) < \pi(i')$, then $j \in G^k$. Therefore, $f^4(N_0, C, G) = Ow(N, v_{C^*}, G)$. In Bergantiños and Gómez-Rúa (2010) we prove that $F(N_0, C, G) = Ow(N, v_{C^*}, G)$. Then, in this case, $f^4(N_0, C, G) = F(N_0, C, G)$, and we have proved above that $F$ satisfies coalitional share of extra costs.

4. $f^4$ fails cost monotonicity. Consider the *mcstp* with groups $(N_0, C, G)$

and $(N_0, C', G)$ where $N = \{1,2\}$, $G = \{\{i\}\}_{i \in N}$, and matrices

$$C = \begin{pmatrix} 0 & 10 & 20 \\ 10 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix} \text{ and } C' = \begin{pmatrix} 0 & 30 & 20 \\ 30 & 0 & 0 \\ 20 & 0 & 0 \end{pmatrix}.$$

In this case, $f^4(N_0, C, G) = (10, 0)$ and $f^4(N_0, C', G) = (0, 20)$.■