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# Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey

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## Abstract

When a firm goes bankrupt, how should its liquidation value be divided among its creditors? This essay is an introduction to the literature devoted to the formal analysis of such problems. We present the rules that are commonly used in practice or discussed in theoretical work. We show how many can be obtained by applying solution concepts developed in cooperative game theory (for bargaining games and for coalitional games). We formulate properties of rules, first when the population of agents is fixed, then when it may vary, compare the rules on the basis of these properties, and search for rules satisfying the greatest number of the properties together. We model the resolution of conflicting claims as strategic games, and extend the model to handle surplus sharing and situations in which the feasible set is specified in utility space. Identifying well-behaved taxation rules is formally identical to identifying rules to reconcile conflicting claims, and all of the results we present can be reinterpreted in that context.

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## 1. Introduction

When a firm goes bankrupt, what is the fair way of dividing its liquidation value among its creditors? This essay is an introduction to the literature devoted to the formal analysis of problems of this kind, which we call “claims problems”. The objective of this literature, which originates in a fundamental paper by O’Neill (1982), is to identify

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well-behaved “rules” for associating with each claims problem a division between the claimants of the amount available.

We first present several rules that are commonly used in practice or discussed in theoretical work. We then formulate a number of appealing properties that one may want rules to satisfy, compare the rules on the basis of these properties, and identify rules satisfying various combinations of the properties. Indeed, the axiomatic method underlies most of the developments on which we report here, and they illustrate the increasingly important role it has been playing in the design of allocation rules. The rapid progress witnessed in the literature on the adjudication of conflicting claims is largely due to researchers being able to draw on the conceptual apparatus and the proof techniques elaborated in the axiomatic analysis of other models, running the gamut from abstract models of game theory and social choice to concrete models of resource allocation. However, we do not limit ourselves to axiomatic studies. We also show how the tools of cooperative game theory, from both the theory of bargaining and the theory of coalitional games, can be used to define rules, and we discuss a variety of strategic approaches.

The best-known rule is the proportional rule, which chooses awards proportional to claims. Proportionality is often taken as the definition of fairness for claims problems, but we will challenge this position and start from more elementary considerations. An important source of inspiration for the research we describe is the Talmud. There, several numerical examples are discussed, and recommendations are made for them that conflict with proportionality. Can these recommendations be rationalized by means of well-behaved rules? Among all existing rules, are there grounds for preferring some to the others? Are there yet other rules that deserve our attention?

Finally, we consider extensions of the model, some covering situations where the amount to divide is more than sufficient to honor all the claims—this is the problem of surplus sharing—and models where the data are specified in utility space and the upper boundary of the feasible set is not restricted to be contained in a hyperplane normal to a vector of ones. We close this introduction by noting that the problem of assessing taxes as a function of incomes, when the total tax to be collected is fixed, is formally identical to the problem of adjudicating conflicting claims. All of the results we present can be re-interpreted in that context, and, more generally, in the context of the assessment of liabilities.

An important question that we will not address is the extent to which the choice of particular division rules affects agents’ incentives to make commitments that one party may in the end be unable to honor. In the context of bankruptcy, these are the incentives to loan and to borrow. In many of the other applications, the parameters of the problems to be solved also result from decisions that agents have made, and whatever rule is used at the division stage will in general have had an effect on these earlier choices. In order to handle these kinds of issues, we would need to embed division rules in a more complete model in which risk-taking, effort, and other variables chosen by agents, such as lenders, borrowers, tax payers, government agencies and others, are explicitly described, stochastic returns to economic activities are factored in, and so on. But the theory developed here, which ignores incentives, is a necessary component of the

comprehensive treatment—it would have to be formulated in a general-equilibrium and game-theoretic framework—that we envision.<sup>1</sup>

## 2. Claims problems and division rules

An amount  $E \in \mathbb{R}_+$  has to be divided among a set  $N$  of agents with claims adding up to more than  $E$ . For each  $i \in N$ , let  $c_i \in \mathbb{R}_+$  denote agent  $i$ 's claim, and  $c \equiv (c_i)_{i \in N}$  the vector of claims. Initially, we take  $N$  to be a finite subset of the set of natural numbers  $\mathbb{N}$ . We sometimes designate by  $n$  the number of elements of  $N$ . Altogether, a *claims problem* is a pair  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$  such that  $\sum c_i \geq E$  (a summation without explicit bounds being understood as applying over all agents).<sup>2,3</sup> Let  $\mathcal{C}^N$  denote the class of all problems. In Section 6, we consider situations in which the population of claimants may vary, and we generalize the model accordingly.

In our primary application,  $E$  is the liquidation value of a bankrupt firm, the members of  $N$  are creditors, and  $c_i$  is the claim of creditor  $i$  against the firm. A closely related application is to estate division: a man dies and the debts he leaves behind, written as the coordinates of  $c$ , are found to add up to more than the worth of his estate,  $E$ . How should the estate be divided? Alternatively, each  $c_i$  could simply be an upper bound on agent  $i$ 's consumption.

When a pair  $(c, E) \in \mathcal{C}^N$  is interpreted as a tax assessment problem, the members of  $N$  are taxpayers, the coordinates of  $c$  are their incomes, and they must cover the cost  $E$  of a project among themselves. The inequality  $\sum c_i \geq E$  indicates that they can jointly afford the project. In this context,  $c_i$  could also be seen as the benefit that consumer  $i$  derives from the project.

Although these various situations can be given the same mathematical description, and the principles relevant to their analysis are essentially the same, the appeal of each particular property of a rule may of course depend on the application. In what follows, we mainly think of the resolution of conflicting claims. Our model is indeed a faithful description of the actual situation faced by bankruptcy courts, for instance. By contrast, the issue of taxation is not always specified by first stating an amount to be collected, perhaps due to the uncertainty pertaining to the taxpayers' incomes. Taxation schedules are usually published first, and the amount collected falls wherever it may, depending upon the realized incomes.<sup>4</sup>

We will search for ways of selecting, for each claims problem, a division between the claimants of the amount available satisfying natural lower and upper bound require-

<sup>1</sup>An important step in that direction is taken by Araujo and Páscoa (2002).

<sup>2</sup>We denote by  $\mathbb{R}_+^N$  the cartesian product of  $|N|$  copies of  $\mathbb{R}_+$  indexed by the members of  $N$ . The superscript  $N$  may also indicate some object pertaining to the set  $N$ . Which interpretation is the right one should be clear from the context. We allow the equality  $\sum c_i = E$  for convenience.

<sup>3</sup>Vector inequalities:  $x \geq y$ ,  $x \geq y$ ,  $x > y$ .

<sup>4</sup>However, we should note that a number of properties of rules that we will use later have been first considered in the context of taxation.

ments. This division is understood as a recommendation for the problem.<sup>5</sup> Formally, a division rule is a function that associates with each  $(c,E) \in \mathcal{C}^N$  a vector  $x \in \mathbb{R}^N$  whose coordinates add up to  $E$  and satisfy the inequalities  $0 \leq x \leq c$ . Such a vector is an awards vector for  $(c,E)$ . Our generic notation for a rule is the letter  $R$ . Given a claims vector, the locus of the awards vector chosen by a rule as the amount to divide varies from 0 to the sum of the claims is the path of awards of the rule for the claims vector.

We will also offer several results concerning a version of the model in which the inequality  $\sum c_i \geq E$  is not imposed, and generalized rules: such a rule may select, for some  $(c,E) \in \mathcal{C}^N$ , an efficient vector  $x$  that does not satisfy the inequalities  $0 \leq x \leq c$ .

### 3. An inventory of rules

We proceed with a presentation of two intriguing problems discussed in the Talmud. The Talmud specifies only a few numerical examples, but the desire to understand them has provided much of the impetus underlying the theoretical efforts described in these pages.

The *contested garment problem* (Fig. 1a): two men disagree on the ownership of a garment, worth 200, say. The first man claims half of it, 100, and the other claims it all, 200. Assuming both claims to be made in good faith, how should the worth of the

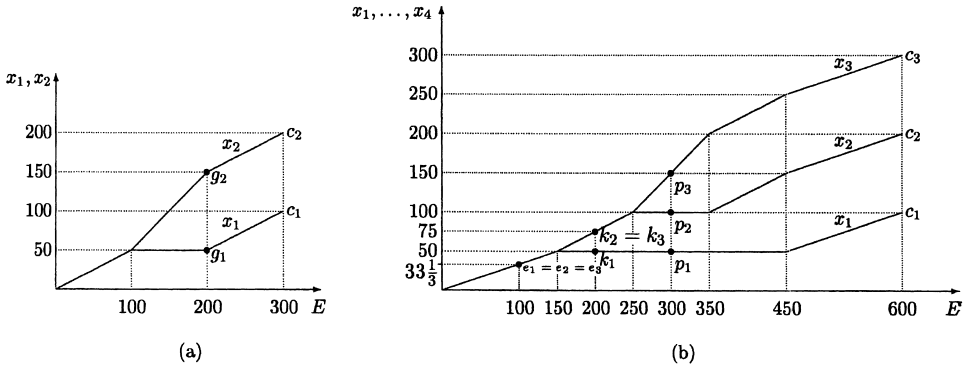


Fig. 1. The Talmud rule. The worth of the estate is measured horizontally; claims and awards are measured vertically. (a) The rule, applied to the claims vector  $(c_1, c_2) \equiv (100, 200)$  of the contested garment problem. The Talmud considers an estate worth 200 and recommends the awards vector  $g \equiv (50, 150)$ . (b) The rule, applied to the claims vector  $(c_1, c_2, c_3) \equiv (100, 200, 300)$  of the estate division problem. For an estate worth 100, the Talmud recommends  $e \equiv (33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$ ; if worth 200, it recommends  $k \equiv (50, 75, 75)$ ; and if worth 300, it recommends  $p \equiv (50, 100, 150)$ .

<sup>5</sup>We limit ourselves to the search for *single-valued* rules, since, for this model, in contrast to a number of other models that are commonly studied, a great variety of interesting rules enjoy this property.

garment be divided between them? The Talmud recommends the awards vector  $g \equiv (50,150)$  (Baba Metzia 2a).<sup>6</sup>

The *estate division problem* (Fig. 1b): a man has three wives whose marriage contracts specify that upon his death they should receive 100, 200, and 300, respectively. The man dies and his estate is found to be worth only 100. The Talmud recommends  $e \equiv (33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$ . If the estate is worth 300, it recommends  $p \equiv (50,100,150)$ . If it is worth 200, it recommends  $k \equiv (50,75,75)$  (Kethubot 93a; the author of this Mishna is Rabbi Nathan).

To clarify the mystery posed by the numbers given as resolutions of these problems, we should first identify a general and natural formula that generates them, and it is only recently that such a formula was found. We give it after introducing several more elementary rules.

(a) The following is a simple division “scenario” for the two-claimant case that delivers the numbers proposed by the Talmud for the contested garment problem (Aumann and Maschler, 1985). Let us refer to the two claimants as agent  $i$  and agent  $j$ . When agent  $i$  claims  $c_i$ , he is essentially conceding to agent  $j$  the amount  $E - c_i$  if this difference is non-negative, and 0 otherwise, that is,  $\max\{E - c_i, 0\}$ . Similarly, claimant  $j$ ’s concession to claimant  $i$  is  $\max\{E - c_j, 0\}$ . Let us then assign to each claimant the amount conceded to him by the other, and divide equally between them what remains, the part that is truly contested. Equal division in this “second round” makes sense since after being revised down by the amounts received in the “first round”, as is very natural, and truncated by the amount that remains available, both claims become equal (the truncation idea is formally introduced and justified below; in many cases, both revised claims are smaller than the remainder, so no truncation is needed).

**Concede-and-divide, CD.** For  $|N| = 2$ . For each  $(c,E) \in \mathcal{C}^N$  and each  $i \in N$ ,

$$CD_i(c,E) \equiv \max\{E - c_j, 0\} + \frac{E - \sum \max_N\{E - c_k, 0\}}{2}.$$

We will discover a number of other ways of thinking about the issue that lead to that same formula in the two-claimant case. However, it is not obvious how to generalize the concede-and-divide scenario itself to the  $n$ -claimant case. The difference for each claimant between the amount to divide and the sum of the claims of the other agents (or 0 if this difference is negative) can certainly still be understood as a “concession” that together they make to him; and it is still natural to assign to him this amount in a first step. However, after being revised down by these amounts, claims need not be equal anymore, even if truncated by the remainder, so equal division at the second step is not as compelling as in the two-claimant case. Besides, it may result in an agent receiving in total more than his claim. In the paragraphs to follow, we therefore explore other ideas.

<sup>6</sup>All references to the relevant passages of the Talmud and medieval literature are taken from O’Neill (1982), Aumann and Maschler (1985), and Dagan (1996).

In each case, we invite the reader to verify that not all of the numbers in the Talmud are accounted for until we reach the “Talmud rule” of Aumann and Maschler (1985).

(b) The most commonly used rule in practice makes awards proportional to claims:

**Proportional rule,  $P$ .** For each  $(c, E) \in \mathcal{C}^N$ ,  $P(c, E) \equiv \lambda c$ , where  $\lambda$  is chosen so that  $\sum \lambda c_i = E$ .

Two other versions of the proportional rule have been proposed. The first one is defined by making awards proportional to the claims truncated by the amount to divide. We refer to it as the *truncated-claims proportional rule*. The other requires first assigning to each claimant the amount that remains if the other claimants have been fully compensated (or 0 if that is not possible). We used this difference above in defining concede-and-divide. It can be seen as a minimum to which the claimant is entitled, and shifting perspectives slightly, we will refer to it by an expression that better reflects that interpretation. Formally, for each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ , let  $m_i(c, E) \equiv \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\}$  be the *minimal right of claimant  $i$  in  $(c, E)$* , and  $m(c, E) \equiv (m_i(c, E))_{i \in N}$ . Then, the rule selects the awards vector at which each claimant receives his minimal right (these payments are feasible); each agent’s claim is revised down to the minimum of: (i) the remainder, and (ii) the difference between his initial claim and his minimal right; finally, the remainder is divided proportionately to the revised claims.

Adjusted proportional rule,  $A$ . (Curiel et al., 1987) For each  $(c, E) \in \mathcal{C}^N$ ,

$$A(c, E) \equiv m(c, E) + P(\{\min\{c_i - m_i(c, E), E - \sum_{j \in N} m_j(c, E)\}\}_{i \in N}, E - \sum_{j \in N} m_j(c, E)).$$

(c) The idea of “equality” underlies many theories of economic justice. The question though is what exactly should be equated, especially when agents are not identical. Here, agents only differ in their claims and equating awards implies ignoring these differences. In particular, an agent may receive more than his claim, which is precluded by our definition of a rule. Our next rule remains close in spirit but it is defined so as to respect these upper bounds on awards. It assigns equal amounts to all claimants subject to no one receiving more than his claim. Although this departure from equality does not seem to be much of a step towards recognizing differences in claims, we will nevertheless provide appealing axiomatic justifications for the rule. It is also an “ingredient” in the definitions of other interesting rules, and we will recognize it in the formulae for Piniles’ and the Talmud rules below. It has been advocated by many authors, including Maimonides (12th Century):

**Constrained equal awards rule,  $CEA$ .** For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$ , where  $\lambda$  is chosen so that  $\sum \min\{c_j, \lambda\} = E$ .

(d) Our next rule (Piniles, 1861) can be understood as resulting from a “double” application of the constrained equal awards rule, the half-claims being used in the formula instead of the claims themselves. First, the constrained equal awards rule is applied to divide the minimum of: (i) the amount available, and (ii) the half-sum of the

claims. If the amount available is less than the half-sum of the claims, we are done. Otherwise, each agent first receives his half-claim; then, the constrained equal awards rule is re-applied to divide the remainder, still using the half-claims in the formula. The rule so obtained accounts for all of the numbers given in the Talmud for the estate division problem.

**Piniles' rule,  $Pin$ .** For each  $(c,E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $Pin_i(c,E) \equiv CEA_i(c/2,E)$  if  $\sum (c_j/2) \geq E$ , and  $Pin_i(c,E) \equiv c_i/2 + CEA_i(c/2,E - \sum (c_j/2))$  otherwise.

(e) Here is another way of implementing the idea of equality (Chun et al., 2001). The formula is inspired by a solution to the problem of fair division when preferences are single-peaked, known as the uniform rule (Sprumont, 1991). As in Piniles, it gives the half-claims a central role, and otherwise, it makes the minimal adjustment in the formula for the uniform rule that guarantees that awards are ordered as claims are.

**Constrained egalitarian rule,  $CE$ .** For each  $(c,E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $CE_i(c,E) \equiv \min\{c_i/2, \lambda\}$  if  $E \leq \sum (c_j/2)$  and  $CE_i(c,E) \equiv \max\{c_i/2, \min\{c_i, \lambda\}\}$  otherwise, where in each case,  $\lambda$  is chosen so that  $\sum CE_i(c,E) = E$ .

(f) An alternative to the constrained equal awards rule is obtained by focusing on the losses claimants incur (what they do not receive), as opposed to what they receive, and choosing the awards vector at which these losses are equal subject to no one receiving a negative amount. It too is discussed by Maimonides (Aumann and Maschler, 1985).<sup>7</sup>

**Constrained equal losses rule,  $CEL$ .** For each  $(c,E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $CEL_i(c,E) \equiv \max\{0, c_i - \lambda\}$ , where  $\lambda$  is chosen so that  $\sum \max\{0, c_i - \lambda\} = E$ .

(g) We now have all that we need to define a rule that finally generates all of the numbers appearing in the Talmud (Aumann and Maschler, 1985; see Fig. 1). As in Piniles' formula, two regimes are defined, depending upon the side of the half-sum of the claims on which the amount to divide falls. For an amount to divide equal to the half-sum of the claims, everyone receives his half-claim. If there is less, the constrained equal awards formula is applied; if there is more, the constrained equal losses formula is; in each case, the half-claims are used in the formula instead of the claims themselves. Alternatively, the rule can be defined by means of an algorithm. First, imagine the amount available increasing from 0 to the half-sum of the claims: the first units are divided equally until each agent receives an amount equal to the smallest half-claim; then, the agent with the smallest claim stops receiving anything for a while, and the next units are divided equally among all others until each of them receives an amount equal to the second smallest half-claim. Then, the agent with the second smallest claim also stops receiving anything for a while, and the next units are divided equally among the other claimants until each of them receives an amount equal to the third smallest

<sup>7</sup>In the context of taxation, this rule is known as "head tax" and the constrained equal awards rule as the "leveling tax".

half-claim . . . . The algorithm proceeds in this way until the amount available is  $\Sigma (c_i/2)$ ; at that point, each agent receives his half-claim. For amounts available greater than  $\Sigma (c_i/2)$ , awards are computed in a symmetric way. Starting from an amount available equal to the sum of the claims, in which case each agent receives his claim, consider shortfalls of increasing sizes: initial shortfalls are divided equally until each agent incurs a loss equal to the smallest half-claim; the loss incurred by the agent with the smallest claim stops at that point and any additional shortfall is born equally by the others, until their common loss is equal to the second smallest half-claim. The algorithm proceeds in this manner until the amount available is  $\Sigma (c_i/2)$ . It is a simple matter to see that, when applied to the two problems in the Talmud, it does yield the numbers given there. Henceforth, we call the rule it defines the *Talmud rule*. The following is a compact definition.<sup>8</sup>

**Talmud rule,  $T$ .** For each  $(c,E) \in \mathcal{C}^N$  and each  $i \in N$ ,

1. If  $\Sigma (c_i/2) \geq E$ , then  $T_i(c,E) \equiv \min\{c_i/2, \lambda\}$ , where  $\lambda$  is chosen so that  $\Sigma \min\{c_i/2, \lambda\} = E$ .
2. If  $\Sigma (c_i/2) \leq E$ , then  $T_i(c,E) \equiv c_i - \min\{c_i/2, \lambda\}$ , where  $\lambda$  is chosen so that  $\Sigma [c_i - \min\{c_i/2, \lambda\}] = E$ .

We refer to the two-claimant version of the Talmud rule as the *contested garment rule*. This rule coincides with concede-and-divide, but as already noted, it is not the only one to do so.<sup>9</sup>

(h) Our inventory of rules is indeed far from being exhausted and our next two rules also coincide with concede-and-divide in the two-claimant case. To define the first one, imagine claimants arriving one at a time, and compensate them fully until money runs out. The resulting awards vector of course depends on the order in which claimants arrive. To remove the unfairness associated with a particular order, take the arithmetic average over all orders of arrival of the awards vectors calculated in this way (O’Neill, 1982). For a formal definition of the resulting rule, let  $\Pi^N$  be the class of bijections from  $N$  into itself.

**Random arrival rule,  $RA$ .** For each  $(c,E) \in \mathcal{C}^N$  and each  $i \in N$ ,

$$RA_i(c,E) \equiv \frac{1}{n!} \sum_{\pi \in \Pi^N} \min \left\{ c_i, \max \left\{ E - \sum_{j \in N, \pi(j) < \pi(i)} c_j, 0 \right\} \right\}.$$

- (i) Another rule is offered by O’Neill (1982) as a generalization of an example found

<sup>8</sup>Of course, the Talmud not offering any example for the case  $E \geq \Sigma (c_i/2)$  when  $n \geq 3$ , we can only speculate as to what it would have recommended then. However, we find the sort of considerations that led Aumann and Maschler (1985) to the interpolation and extrapolation they define very compelling, and this is why we refer to the rule they propose as the Talmud rule. Moreover, their formula is in agreement with another numerical example in the Talmud that they discuss.

<sup>9</sup>We have  $T(c,E) = CEA(c/2,E)$  if  $\Sigma (c_i/2) \geq E$  and  $T(c,E) = (c/2) + CEL[c/2,E - \Sigma (c_i/2)]$  otherwise.

in Ibn Ezra (12th Century) and of an incompletely specified rule due to Rabad (12th Century). The problem discussed by Ibn Ezra is that of dividing an estate worth 120 among four sons whose claims are 30, 40, 60, and 120. He recommends  $\frac{30}{4}$  for the first son,  $\frac{30}{4} + \frac{10}{3}$  for the second son,  $\frac{30}{4} + \frac{10}{3} + \frac{20}{2}$  for the third son, and  $\frac{30}{4} + \frac{10}{3} + \frac{20}{2} + \frac{60}{1}$  for the fourth son.

Rabad's suggestion, which gives Ibn Ezra's numbers in his particular application, is defined for problems such that the estate is worth no more than the greatest claim (Aumann and Maschler, 1985): if the estate is worth less than the smallest claim, it is divided equally; as its worth increases from the smallest to the second smallest claim, the agent with the smallest claim continues to receive  $1/n$  of his claim and the remainder is divided equally among the other claimants. In general, when the worth of the estate increases from the  $k$ th smallest claim to the  $(k + 1)$ th smallest claim, the amounts received by the agents with the  $k$  smallest claims do not change, and the remainder is divided equally among the other claimants.

Here is O'Neill's proposal for general problems. Instead of thinking of claims abstractly, imagine that the amount to divide is composed of individual and distinct "units", and that each claim is on specific units. Then, distribute claims over these units so as to maximize the fraction of the estate claimed by exactly one claimant, and subject to that, so as to maximize the fraction claimed by exactly two claimants, and so on; finally, for each unit separately, apply equal division among all agents claiming it.

**Minimal overlap rule, MO.** Claims on specific parts of the amount available, or "units", are arranged so that the number of units claimed by exactly  $k + 1$  claimants is maximized, given that the number of units claimed by  $k$  claimants is maximized, for  $k = 1, \dots, n - 1$ . Then, for each unit, equal division prevails among all agents claiming it. Each claimant collects the partial compensations assigned to him for each of the units that he claims.

The arrangement of claims solving this lexicographic maximization is unique up to inessential relabelling of units. A claim greater than the amount available is equivalent to a claim equal to the amount available, and if there is at least one such claim, the solution consists in nesting the claims. If not, representing the amount available as an interval  $[0, E] \subset \mathbb{R}_+$ , the solution is obtained by finding  $t \in \mathbb{R}_+$  such that each agent  $i \in N$  claims the interval  $[0, \min\{c_i, t\}]$ —this takes care of the agents whose claims are at most  $t$ —and each of the agents whose claim is larger than  $t$  claims  $[0, t]$  plus a subinterval of  $[t, E]$ ; these subintervals do not overlap and together they cover  $[t, E]$  (therefore  $t$  satisfies  $\sum_{\{i \in N : c_i > t\}} (c_i - t) = E - t$ ).<sup>10</sup> As is easily verified, the rule so defined generates Ibn

<sup>10</sup>O'Neill (1982) defines a method based on the following stochastic scenario: agents randomly make claims on specific parts of the estate, the total amount claimed by each agent being equal to his claim; for each part of the estate, equal division prevails among all agents claiming it. Unfortunately, this method may not attribute the whole estate (it is not *efficient*, as formally defined below). Moreover, when claims are compatible, it need not award to each agent an amount equal to his claim. In order to recover *efficiency*, O'Neill suggests taking the amounts awarded by the method as a starting point, and applying the method again to distribute the remainder. Then, an agent may get more than his claim. To remedy this problem, adjust down each agent's claim by what he receives initially. Repeat the process and take the limit.

Ezra’s numbers in the example he considers (nesting applies). Also, in the two-claimant case, it coincides with concede-and-divide.

(j) We will close our inventory with families of rules. These families tie together several of the rules that we have already listed but each has infinitely many members. The first one, called the *ICI family* (for Increasing-Constant-Increasing, an expression reflecting the evolution of each claimant’s award as a function of the amount to divide), can be seen as generalizing the Talmud rule. The only difference is that the points at which agents temporarily stop receiving additional units and the points at which they are invited back in are allowed to depend on the claims vector. Otherwise, agents leave and come back in the same order as for the Talmud rule; also, any two agents who are present at any given time receive equal shares of any increment.

Here is the formal definition (Thomson, 2000). Let  $\mathcal{G}^N$  be the family of lists  $G \equiv \{E_k, F_k\}_{k=1}^{n-1}$ , where  $n \equiv |N|$ , of real-valued functions of the claims vector, satisfying for each  $c \in \mathbb{R}_+^N$ , the following relations, which we call the *ICI relations*. These relations are imposed to guarantee that, at the end of the process just described, each agent is fully compensated. Let  $C \equiv \sum c_i$

$$\begin{array}{rccccccc}
 & & \frac{E_1(c)}{n} & + & \frac{C - F_1(c)}{n} & = & c_1 \\
 c_1 & + & \frac{E_2(c) - E_1(c)}{n - 1} & + & \frac{F_1(c) - F_2(c)}{n - 1} & = & c_2 \\
 \vdots & + & \vdots & + & \vdots & = & \vdots \\
 c_{k-1} & + & \frac{E_k(c) - E_{k-1}(c)}{n - k + 1} & + & \frac{F_{k-1}(c) - F_k(c)}{n - k + 1} & = & c_k \\
 \vdots & + & \vdots & + & \vdots & = & \vdots \\
 c_{n-1} & + & \frac{-E_{n-1}(c)}{1} & + & \frac{F_{n-1}(c)}{1} & = & c_n.
 \end{array}$$

**ICI rule relative to**  $G \equiv \{E_k, F_k\}_{k=1}^{n-1} \in \mathcal{G}^N, R^G$ . For each  $c \in \mathbb{R}_+^N$ , the awards vector is given as the following function of the amount available  $E$ , as it varies from 0 to  $\sum c_i$ . As  $E$  increases from 0 to  $E_1(c)$ , equal division prevails; as it increases from  $E_1(c)$  to  $E_2(c)$ , claimant 1’s award remains constant, and equal division of each new unit prevails among the other claimants. As  $E$  increases from  $E_2(c)$  to  $E_3(c)$ , claimants 1 and 2’s awards remain constant, and equal division of each new unit prevails among the other claimants, and so on. This process goes on until  $E$  reaches  $E_{n-1}(c)$ . The next units go to claimant  $n$  until  $E$  reaches  $F_{n-1}(c)$ , at which point equal division of each new unit prevails among claimants  $n$  and  $n - 1$ . This goes on until  $E$  reaches  $F_{n-2}(c)$ , at which point equal division of each new unit prevails among claimants  $n$  through  $n - 2$ . The process continues until  $E$  reaches  $F_1(c)$ , at which point claimant 1 re-enters the scene and equal division of each new unit prevails among all claimants.

The constrained equal awards, constrained equal losses, Talmud, and minimal overlap rules belong to the family. A one-parameter subfamily of the ICI family, indexed by  $\theta \in [0,1]$ , is defined by having the  $k$ th agent, as determined by the order of claims, drop out when his award reaches the fraction  $\theta$  of his claim, and having him return when his

loss reaches the fraction  $(1 - \theta)$  of his claim (Moreno-Ternero and Villar, 2001). The constrained equal awards rule is obtained for  $\theta = 1$ , the constrained equal awards rule for  $\theta = 0$ , and the Talmud rule for  $\theta = 1/2$ .

A “reverse” family of the ICI family can be defined in which the order in which agents are handled is reversed. Agents arrive in the order of decreasing claims and they drop out in the order of increasing claims, when fully compensated (Thomson, 2000). As for the ICI family, any two agents who are present at any given time receive equal shares of each new unit. As a function of the amount to divide, each agent’s award is constant in some initial interval, then increases, then is constant again. This *CIC family* (for Constant-Increasing-Constant) contains the constrained equal awards and constrained equal losses rules.

(k) Each member of the third family (Young, 1987a) is indexed by a function  $f : \mathbb{R}_+ \times [a,b] \rightarrow \mathbb{R}_+$ , where  $[a,b] \subset ]-\infty, +\infty[$  that is continuous, nowhere decreasing in its second argument, and such that for each  $\bar{c} \in \mathbb{R}_+$ ,  $f(\bar{c},a) = 0$  and  $f(\bar{c},b) = \bar{c}$ . Let  $\Phi$  be the family of these functions. They can be interpreted as telling us how much each agent should receive in order to experience a certain welfare if his claim has a certain value. The resource is then divided so as to ensure that all agents experience equal welfares. At this point, the function  $f$  is not required to satisfy any other requirement, but each property required of the resulting rule will translate into restrictions on  $f$ .

**Parametric rule of representation**  $f \in \Phi, R^f$ . For each  $N \in \mathcal{N}$  and each  $(c,E) \in \mathcal{C}^N$ ,  $R^f(c,E)$  is the awards vector  $x$  such that there exists  $\lambda \in [a,b]$  for which, for each  $i \in N$ ,  $x_i = f(c_i, \lambda)$ .

Many rules belong to the family: the proportional, constrained equal awards, constrained equal losses, Piniles, Talmud, and constrained egalitarian rules are members. Fig. 2a and b give parametric representations of the proportional and Talmud rules, the latter when an upper bound on claims exists,  $c_{\max}$ .<sup>11</sup>

In the context of taxation, the following parametric rules have also been discussed: for *Stuart’s rule*,  $f(c_i, \lambda) \equiv \max\{c_i - c_i^{1-\lambda}, 0\}$  and for *Cassel’s rule*,

$$f(c_i, \lambda) \equiv \frac{c_i^2}{c_i + (1/\lambda)} \text{ for } 0 \leq \lambda \leq \infty.$$

#### 4. Relating division rules and solution concepts of the theory of cooperative games

In this section, we exhibit interesting relations between certain division rules and various solution concepts of the theory of cooperative games. For this theory to be applicable, we need first to define a formal way of associating with each claims problem

<sup>11</sup>Imposing this bound restricts somewhat the scope of the rule but it permits a very simple (piecewise linear) representation (Chun et al., 2001). See Young (1987a) for a representation without the upper bound.

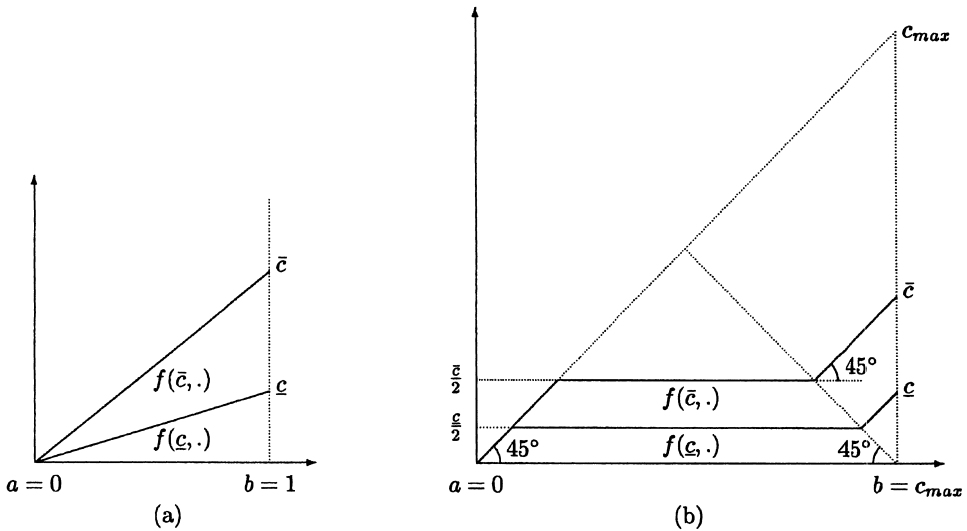


Fig. 2. Parametric representations of two rules. (a) Proportional rule: the domain of the representation depicted here is the interval  $[0,1]$ . The schedules are segments through the origin, of slopes equal to claims. (b) Talmud rule: we assume for simplicity that there is a maximal value that a claim can take,  $c_{max}$ . The domain of the representation depicted here is the interval  $[0, c_{max}]$ . The schedule relative to a typical claim  $c$  follows the  $45^\circ$  line up to the point  $(c/2, c/2)$ , continues horizontally until it meets the line of slope  $-1$  emanating from  $(c_{max}, 0)$ , then again, follows a line of slope  $1$  until it reaches  $(c_{max}, c)$ .

a cooperative game. Two main classes of such games have been studied, bargaining games and coalitional games, and accordingly, we establish two kinds of relations.

#### 4.1. Bargaining solutions

A bargaining game is a pair  $(B, d)$ , where  $B$  is a subset of  $\mathbb{R}^N$ , the  $n$ -dimensional Euclidean space, and  $d$  is a point of  $B$ . The set  $B$ , the *feasible set*, consists of all utility vectors attainable by the group  $N$  by unanimous agreement, and  $d$ , the *disagreement point*, is the utility vector that results if they fail to reach an agreement. A bargaining solution is a function defined on a class of bargaining games that associates with each game in the class a unique point in the feasible set of the game. The following are important examples. The *egalitarian solution* (Kalai, 1977) selects the maximal point of  $B$  at which all agents' utility gains from  $d$  are equal. The *lexicographic egalitarian solution* (Imai, 1983) selects the point of  $B$  at which these gains are maximal in the lexicographic (maximin) order.<sup>12</sup> The *Kalai–Smorodinsky solution* (Kalai and Smorodinsky, 1975) selects the maximal point of  $B$  on the segment connecting  $d$  to the “ideal point of  $(B, d)$ ,” the point whose  $i$ th coordinate, for each  $i \in N$ , is the maximal

<sup>12</sup>Given  $x, y \in \mathbb{R}^N$ , let  $\tilde{x}$  designate the vector obtained from  $x$  by rewriting its coordinates in increasing order,  $\tilde{y}$  being similarly defined. We say that “ $x$  is greater than  $y$  in the lexicographic (maximin) order” if either  $[\tilde{x}_1 > \tilde{y}_1]$ , or  $[\tilde{x}_1 = \tilde{y}_1 \text{ and } \tilde{x}_2 > \tilde{y}_2]$ , or, more generally, for some  $k \in [1, \dots, n]$ ,  $[\tilde{x}_1 = \tilde{y}_1, \dots, \tilde{x}_{k-1} = \tilde{y}_{k-1}, \text{ and } \tilde{x}_k > \tilde{y}_k]$ .

utility agent  $i$  can obtain subject to the condition that for each  $j \in N \setminus \{i\}$ , agent  $j$  receives at least  $d_j$ . The *Nash solution* (Nash, 1950) selects the point maximizing the product of utility gains from  $d$  among all points of  $B$  dominating  $d$ . Given a vector of weights  $\alpha \in \text{int}\Delta^N$  (where  $\Delta^N$  denotes the unit simplex in  $\mathbb{R}^N$  and  $\text{int}$  its interior), the *weighted Nash solution with weights  $\alpha$*  selects the point of  $B$  at which the product  $\prod (x_i - d_i)^{\alpha_i}$  is maximized among all points of  $B$  dominating  $d$ . The *extended equal losses solution* (Bossert, 1993, in a contribution building on the equal losses solution of Chun, 1988b) selects the maximal feasible point at which the losses from the ideal point of all agents are equal, except that any agent who would then end up with a utility smaller than his disagreement utility is assigned his disagreement utility instead.

The most natural way to associate a bargaining game with a claims problem is to take as feasible set the set of all non-negative vectors dominated by the claims vector and whose coordinates do not add up to more than the amount available, and to choose the origin as disagreement point. This makes sense since we require that a rule should never assign to any agent more than his claim. However, a rule that satisfies this requirement could be responsive to changes in claims that do not affect the associated bargaining game as just defined (the proportional rule is an example), and one could argue that too much information is lost in the passage from claims problems to bargaining games. Nevertheless, given a claims problem  $(c, E) \in \mathcal{C}^N$ , its *associated bargaining game* is the game with feasible set  $B(c, E) \equiv \{x \in \mathbb{R}^N : \sum x_i \leq E, 0 \leq x_i \leq c_i\}$ , and with disagreement point  $d \equiv 0$ . Note that  $d$  does not depend on  $(c, E)$ .<sup>13</sup>

This definition is illustrated in Fig. 3 for two examples. For the first example, the vector of equal awards does not belong to the undominated boundary of the feasible set, but for the second it does.

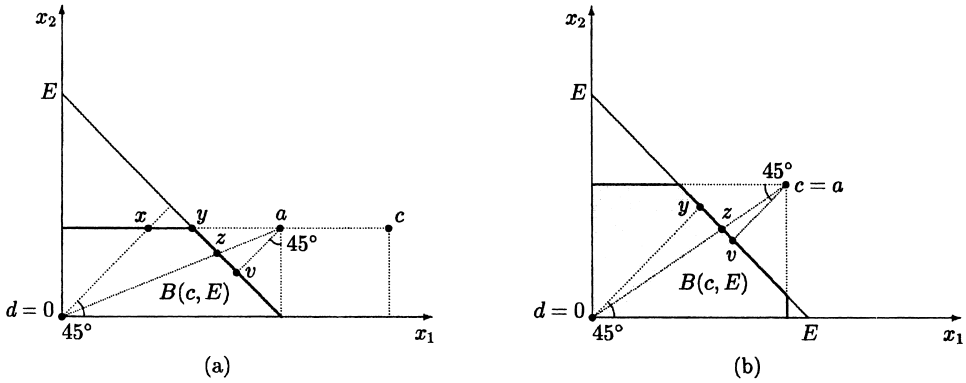
In the classical theory of bargaining, the feasible set is allowed to be an arbitrary compact and convex set, but here we have the special case of a feasible set whose efficient boundary is a subset of a plane normal to a vector of ones. An extension of the model accommodating more general shapes is discussed in Section 8.2.

If for each claims problem, the recommendation made by a given rule coincides with the recommendation made by a given bargaining solution when applied to the associated bargaining game, *the rule corresponds to the solution*. Our first theorem describes several such correspondences.

**Theorem 1.** *The following correspondences between division rules and bargaining solutions hold:*

1. *the constrained equal awards rule and the Nash bargaining solution (Dagan and Volij, 1993);*
2. *the constrained equal awards rule and the lexicographic egalitarian solution;*
3. *the proportional rule and the weighted Nash solution with the weights chosen equal to the claims (Dagan and Volij, 1993);*

<sup>13</sup>An alternative specification of the disagreement point is possible that does, namely the vector of minimal rights (Dagan and Volij, 1993).



$$\begin{aligned}
 y &= CEA(c, E) = N(B(c, E), d) = E^L(B(c, E), d) \\
 z &= P^t(c, E) = K(B(c, E), d) = N^c(B(c, E), d) \\
 v &= CEL^t(c, E) = XEL(B(c, E), d)
 \end{aligned}$$

Fig. 3. Claims problems and their associated bargaining games. Key for bargaining solutions:  $N$ , Nash solution;  $K$ , Kalai–Smorodinsky solution;  $E^L$ , lexicographic egalitarian solution;  $N^c$ , weighted Nash solution with weights  $c$ ;  $XEL$ , extended equal losses solution. The shaded region  $B(c, E)$  represents the set of vectors  $x \in \mathbb{R}_+^2$  such that  $\sum x_i \leq E$  and that are dominated by the claims vector. The region is taken as the feasible set of the bargaining game associated with  $(c, E)$ . The disagreement point  $d$  is placed at the origin. The egalitarian bargaining solution selects the maximal point of  $(B(c, E), d)$  of equal coordinates. At such a point,  $x$ , the whole estate need not be divided, as shown in panel (a). In the two-agent case, the extended equal losses bargaining solution simply selects the maximal point of  $(B(c, E), d)$  at which losses from the ideal point of  $(B(c, E), d)$ , the point  $a$ , are equal (the point  $v$ ).

4. *the truncated-claims proportional rule and the Kalai–Smorodinsky solution (Dagan and Volij, 1993);<sup>14</sup>*
5. *the truncated-claims constrained equal losses rule (obtained from the constrained equal losses rule by truncating claims by the amount to divide) and the extended equal losses bargaining solution.*

The recommendations made by various division rules and the bargaining solutions to which they correspond are indicated in Fig. 3 for two examples.

Theorem 1 establishes useful links between the theory developed here for the adjudication of conflicting claims and the theory of bargaining, but one should perhaps not attach too much importance to any particular one of them. Indeed, since the bargaining games associated with claims problems constitute a very narrow and particularly simple subclass of the class of bargaining games traditionally studied, it follows that bargaining solutions that in general produce different payoff vectors often coincide on this subclass. This phenomenon is illustrated by the fact that the constrained

<sup>14</sup>Dagan and Volij (1993) also show that the adjusted proportional rule corresponds to the Kalai–Smorodinsky solution applied to the problem in which the disagreement point is set equal to the vector of minimal rights instead of the origin.

equal awards rule corresponds to both the Nash solution and the lexicographic egalitarian solution.<sup>15</sup>

Another conversion of claims problems into bargaining games is possible, however, in which the claims point remains as separate data. The relevant concept is then the generalization of the notion of a bargaining game obtained by adding a claims point (Chun and Thomson, 1992, study these problems under the name of “bargaining problems with claims”. See Section 8.2 for further discussion). Then, the proportional rule corresponds to the solution also called “proportional” in that theory.

#### 4.2. Solutions to coalitional games

We now turn to the richer class of coalitional games. Such games are formal representations of situations in which all groups, or *coalitions* (and not just the group of the whole), can achieve something. Formally, a (transferable utility) *coalitional game* is a vector  $v \equiv (v(S))_{S \subseteq N} \in \mathbb{R}^{2^{|N|-1}}$ , where for each coalition  $\emptyset \neq S \subseteq N$ ,  $v(S) \in \mathbb{R}$  is the *worth of S*. This number is interpreted as what the coalition “can obtain on its own” or “can guarantee itself”. A *solution* is a mapping that associates with each game  $v$  in some admissible class a *payoff vector*, a point in  $\mathbb{R}^N$  whose coordinates add up to  $v(N)$ , a property to which we will also refer as “efficiency”.

In order to be able to apply the solutions discussed in the theory of coalitional games, we need a procedure for associating with each claims problem such a game. The most common one is to set the worth of each coalition  $S$  equal to the difference between the amount available and the sum of the claims of the members of the complementary coalition,  $N \setminus S$ , if this difference is non-negative, and 0 otherwise. Using terminology introduced earlier, the difference can be understood as what the complementary coalition “concedes” to  $S$ . It is certainly what  $S$  can secure without going to court. Formally, given a claims problem  $(c, E) \in \mathcal{C}^N$ , its *associated coalitional game* (O’Neill, 1982) is the game  $v(c, E) \in \mathbb{R}^{2^{|N|-1}}$  defined by setting for each  $\emptyset \neq S \subseteq N$ ,  $v(c, E)(S) \equiv \max\{E - \sum_{N \setminus S} c_i, 0\}$ .

Note that our definition is in agreement with the standard manner in which TU games are constructed to represent conflicts, namely by defining the worth of a coalition as a guaranteed amount. If the worth of a coalition is interpreted instead as the amount the coalition “can expect to receive”, the definition is somewhat pessimistic. However, the bias being systematic across coalitions, we might still feel that the resulting game appropriately summarizes the situation.<sup>16</sup>

<sup>15</sup>The reader may wonder why a solution that is *scale invariant* (invariant with respect to positive linear transformations, operating on each coordinate separately, of their utilities), such as the Nash solution, can coincide with a solution that involves utility comparisons, such as the lexicographic egalitarian solution. The answer is simply that the subclass of bargaining games associated with claims problems is not rich enough for scale transformations in which the scale coefficients differ between agents to ever be applicable.

<sup>16</sup>A coalitional game is a point in a space of considerably greater dimension than a claims problem. One could argue that the passage from claims problems to coalitional games involves an undesirable increase in dimensionality.

The game  $v(c,E)$  is convex<sup>17</sup> (Aumann and Maschler, 1985). Therefore, its core<sup>18</sup> is non-empty. In fact, this set is simply the set of awards vectors of  $(c,E)$  (recall that these are the vectors  $x \in \mathbb{R}^N$  such that  $\sum x_i = E$  and  $0 \leq x_i \leq c_i$ ).

If for each claims problem, the recommendation made by a given division rule coincides with the recommendation made by a given solution to coalitional games when applied to the associated coalitional game, once again we say that *the rule corresponds to the solution*. Just as we saw for bargaining solutions, a number of correspondences exist between the division rules introduced in Section 3 and solutions to coalitional games.

In the two-claimant case, for each  $i \in N$ ,  $v(\{i\})$  is equal to  $E - c_j$ , where  $j \neq i$ , if this difference is non-negative, and 0 otherwise;<sup>19</sup> the worth of the grand coalition is equal to  $E$ . Dividing equally the amount that remains when each claimant  $k$  is first paid  $v(\{k\})$  is what virtually all standard solutions to coalitional games recommend. It also corresponds to the recommendation made by concede-and-divide. Then, agent  $i$ 's payoff is

$$x_i = \max\{E - c_j, 0\} + \frac{1}{2}[E - \sum \max\{E - c_k, 0\}].$$

The first correspondence we describe for the  $n$ -claimant case involves the random arrival rule and the solution to coalitional games introduced by Shapley (1953). Most convenient here is the “random arrival” definition of this solution. The *Shapley value of player  $i \in N$  in the game  $v \in \mathbb{R}^{2^{|N|-1}}$*  is the expected amount by which his arrival changes the worth of the coalition consisting of all the players who have arrived before him, assuming all orders of arrival to be equally likely, with the convention that the worth of the empty set is 0. Recall that  $\Pi^N$  is the class of bijections from  $N$  into itself. Then, for each  $v \in \mathbb{R}^{2^{|N|-1}}$  and each  $i \in N$ ,

$$Sh_i(v) \equiv \frac{1}{|N|!} \sum_{\pi \in \Pi^N} [v(\{j \in N : \pi(j) < \pi(i)\} \cup i) - v(\{j \in N : \pi(j) < \pi(i)\})].^{20}$$

Another important solution to coalition games is the *prenucleolus* (Schmeidler, 1969). First, define the “dissatisfaction of a coalition at a proposed payoff vector” to be the difference between its worth and the sum of the payoffs to its members. Then the prenucleolus is obtained by performing the following sequence of minimizations: first, identify the efficient vectors at which the dissatisfaction of the most dissatisfied coalition

<sup>17</sup>This means that the contribution of a player to any coalition is at least as large as his contribution to any subcoalition of it.

<sup>18</sup>This is the set of efficient payoff vectors such that each coalition receives at least its worth: more precisely, the core of  $v \in \mathbb{R}^{2^{|N|-1}}$  is the set of payoff vectors  $x \in \mathbb{R}^N$  such that  $\sum x_i = v(N)$  and for each  $S \subset N$ ,  $\sum_S x_i \geq v(S)$ .

<sup>19</sup>What we called the minimal right of a claimant is simply the worth of the “coalition” consisting only of that claimant.

<sup>20</sup>O’Neill (1982) discusses a method of step-by-step adjustments of particular historical significance (under the name of “recursive completion”) that produces the Shapley value.

is the smallest; among the minimizers, identify the vectors at which the dissatisfaction of the second most dissatisfied coalition is the smallest, and so on.<sup>21</sup>

The *Dutta–Ray solution* selects, for each convex game, the payoff vector in the core that is Lorenz-maximal<sup>22</sup> (Dutta and Ray, 1989).

Finally, consider the  $\tau$ -value (Tijs, 1981). It is defined by first calculating a “maximal payoff” and a “minimal payoff” for each player; then, it chooses the efficient payoff vector that lies on the segment connecting the vector of minima to the vector of maxima. For each  $v \in \mathbb{R}^{2^{|N|-1}}$  and each  $i \in N$ ,  $M_i(v) \equiv v(N) - v(N \setminus \{i\})$  and

$$m_i(v) \equiv \max_{S \subseteq N, i \in S} \left( v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right).$$

Then,  $\tau(v) \equiv \lambda M(v) + (1 - \lambda)m(v)$ , where  $\lambda$  is chosen so as to obtain efficiency.<sup>23</sup>

The next theorem links four of the division rules introduced earlier to solutions to coalitional games.

**Theorem 2.** *The following correspondences between division rules and solutions to coalitional games hold:*

1. *the random arrival rule and the Shapley value (O’Neill, 1982);*
2. *the Talmud rule and the prenucleolus (Aumann and Maschler, 1985);<sup>24</sup>*
3. *the constrained equal awards rule and the Dutta–Ray solution (Dutta and Ray, 1989);*
4. *the adjusted proportional rule and the  $\tau$ -value (Curiel et al., 1987).*

Of course, not every division rule corresponds to some solution for coalitional games. A necessary and sufficient condition for such a correspondence to exist is that the rule depends only on the truncated claims and the amount available (Curiel et al., 1987).<sup>25</sup>

We close this section by noting that other ways of associating a coalitional game to a claims problem are conceivable. Indeed, we have already pointed out that our earlier definition reflects a rather pessimistic assessment of what a coalition can expect. An alternative assessment, this time optimistic, of the situation, leads to the formula  $w(c, E)(S) \equiv \min\{\sum_S c_i, E\}$  for each  $S \subseteq N$ . The resulting game  $w(c, E)$  is studied by Driessen (1995). Here too, in spite of the bias of the definition, the fact that it is systematic across coalitions gives us the hope that the game still provides a useful

<sup>21</sup>On the subclass of convex games, the prenucleolus coincides with the “nucleolus” and, more interestingly, as this solution is usually multi-valued, with the “kernel”.

<sup>22</sup>Given  $x$ , recall (footnote 12) that  $\tilde{x}$  denotes the vector obtained from  $x$  by rewriting its coordinates in increasing order. Given  $x$  and  $y \in \mathbb{R}^N$  with  $\sum x_i = \sum y_i$ , we say that  $x$  is *greater than  $y$  in the Lorenz ordering* if  $\tilde{x}_1 \geq \tilde{y}_1$  and  $\tilde{x}_1 + \tilde{x}_2 \geq \tilde{y}_1 + \tilde{y}_2$ , and  $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \geq \tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3$ , and . . . , with at least one strict inequality.

<sup>23</sup>For a convex game  $v$ ,  $M(v)$  dominates  $m(v)$ .

<sup>24</sup>Lee (1992) and Benoît (1997) give alternative proofs of this result.

<sup>25</sup>This means that  $R(c, E)$  can be written as  $\tilde{R}(\min\{c_i, E\})_{i \in N}, E$  for some function  $\tilde{R}$ .

summary of the situation. Driessen's results should reassure us with respect to the sensitivity of our conclusions to how the game is specified: the core of  $v(c,E)$  actually coincides with the "anticore" of  $w(c,E)$ ,<sup>26</sup> and their Shapley values and nucleoli coincide.<sup>27</sup>

## 5. Properties of rules: the fixed-population case

In this section and the next, we formulate properties of rules and examine how permissive or restrictive they are. We start with the ones we consider the most natural. We continue with properties that one may or may not want to impose depending upon the range of situations to be covered and upon the legal or informational constraints that have to be respected.

### 5.1. Basic properties

*Feasibility* is simply the requirement that for each problem, the sum of the awards should not exceed the amount available, and *efficiency* the stronger requirement that the entire amount available should be allocated. We have incorporated *efficiency* in the definition of a rule. It is obvious that we cannot distribute more than there is, but conceivably, we could distribute less, and if this allowed recovering other properties of interest, we might be willing to consider the possibility. Nevertheless, we will insist on equality between sum of awards and amount available, as this entails no loss.<sup>28</sup>

Next are two requirements placing bounds on awards. Both are very natural and we have also incorporated them in the definition of a rule. *Non-negativity* gives a lower bound: each claimant should receive a non-negative amount. *Claims boundedness* gives an upper bound: each agent should receive at most his claim.

Another lower bound requirement is that each claimant should receive at least the difference between the amount available and the sum of the claims of the other claimants if this difference is non-negative, and 0 otherwise. Recall that we referred to this quantity as the claimant's "minimal right".<sup>29</sup>

<sup>26</sup>This is the set of vectors  $x \in \mathbb{R}^N$  such that  $\sum x_i = v(N)$  and for each  $S \subseteq N$ ,  $\sum_S x_i \leq w(c,E)(S)$ . In fact, Driessen (1995) shows that the two games  $v$  and  $w$  are duals of each other in a formal sense.

<sup>27</sup>A third formulation, providing a compromise between the pessimistic and optimistic outlooks leading to  $v(c,E)$  and  $w(c,E)$ , respectively, simply consists in taking the average  $\{[v(c,E) + w(c,E)]/2\}$ . The analysis of the resulting game remains to be carried out, although it is easy to see that in the two-claimant case, its core is a singleton, and this singleton coincides with the recommendation made by concede-and-divide. Finally, O'Neill (1982) suggests having each coalition and its complement play a two-player strategic game of "positioning of claims" as described in Section 7 below, and defines the worth of a coalition as what it receives at the (essentially unique) Nash equilibrium of the game (see Theorem 20).

<sup>28</sup>It may seem that one should never want to allocate less than is available, but in other settings, such an option has proved extremely useful. In the context of public good decision, for example, the so-called Clarke–Groves mechanism succeeds in eliciting truthful information about agents' preferences only because it fails budget balance. (Recall the cost allocation interpretation of our model.)

<sup>29</sup>Using the language of the theory of cooperative games, this property could be called "individual rationality".

**Respect of minimal rights.** For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,  $R_i(c, E) \geq \max\{E - \sum_{N \setminus \{i\}} c_j, 0\}$ . *Respect of minimal rights* is a consequence of *efficiency*, *non-negativity*, and *claims boundedness* together.

Fully compensating agents with small claims is of course relatively easier and, in some circumstances, it may be tempting to do so first. Which criterion one should adopt to decide how small a claim should be for its owner to deserve this preferential treatment is a matter of judgment, but one interesting critical value of a claim is obtained as follows: substitute it for the claim of any other agent whose claim is higher, and check whether there would then be enough to compensate everyone (Herrero and Villar, 2001b; they use the term “sustainability”). Of all of the rules that we have seen, only the constrained equal awards rule satisfies the resulting requirement:

**Conditional full compensation.** For each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ , if  $\sum_{j \in N} \min\{c_j, c_i\} \leq E$ , then  $R_i(c, E) = c_i$ .

An alternative choice for a critical value of a claim below which full compensation could be required is simply  $E/n$  (Herrero and Villar, 2001b; they refer to the resulting property as “exemption”). If  $|N| = 2$ , the two properties are equivalent.

Symmetrically, one could adopt the viewpoint that if a claim is too small, one should not bother assigning anything to its owner. In the context of bankruptcy for example, the objective would be to give priority to agents who have risked relatively greater amounts. In the context of taxation, exempting agents with lower incomes is a feature of almost all real-world tax laws. We refer to this property as *conditional null compensation* (Herrero and Villar, 2001b; they use the expression “independence of residual claims” and “exclusion” for the counterpart of “exemption”, obtained by comparing claims to the deficit per capita).

Next, we require that the awards to agents whose claims are equal should be equal.

**Equal treatment of equals.** For each  $(c, E) \in \mathcal{C}^N$  and each  $\{i, j\} \subseteq N$ , if  $c_i = c_j$ , then  $R_i(c, E) = R_j(c, E)$ .

The requirement is not always justified and, in fact, it is often violated in practice. In actual bankruptcy proceedings, for example, some claims (for instance, secured claims) have higher priority than others (such as unsecured claims). To allow differential treatment of otherwise identical agents, we can enrich the model and explicitly introduce “priority” parameters. A *claims problem with priorities* is a list  $(c, \preceq, E)$  where  $(c, E) \in \mathcal{C}^N$  and  $\preceq$  is a complete and transitive binary relation on  $N$ . Given  $i \in N$ , the equivalence class containing  $i$  is his *priority class*: two agents in the same priority class are treated differently only to the extent that their claims differ, but agents in different priority classes are usually treated differently even when their claims are equal. Priorities classes are handled in succession, nothing being assigned to any class before the claims of all members of the higher classes are fully satisfied.

All rules can be adapted to accommodate priorities. To illustrate, and taking the

proportional rule as point of departure, we would make awards proportional to claims within each priority class. In fact, this is commonly done (Aggarwal, 1992).

Alternatively, and somewhat more flexibly, we could define a *claims problem with weights* to be a list  $(c, \alpha, E)$ , where  $(c, E) \in \mathcal{C}^N$  and  $\alpha \in \text{int}\Delta^N$  is a point in the interior of the unit simplex of  $\mathbb{R}^N$  indicating what could be called the “relative” priorities (as opposed to the “absolute” priorities of the previous paragraph) that should be given to agents. Most rules can easily be adapted to this setting too. For instance, to obtain a weighted version of the proportional rule, make awards proportional to the vector of weighted claims (each claim is multiplied by the weight assigned to the agent holding it), subject to no one receiving more than his claim. For a weighted version of the constrained equal awards rule, perform the division proportionally to the claimants’ weights subject to no one receiving more than his claim. For a weighted version of concede-and-divide, calculate the concessions as in the initial definition, but in the second stage, perform the division proportionally to the claimants’ weights.<sup>30</sup>

Priorities and weights can be combined. Divide agents into priority classes to be handled in succession; within each class, use weights to deflect the awards vector in the desired direction.

A strengthening of *equal treatment of equals* is the requirement that the identity of agents should not matter. The chosen awards vector should depend only on the list of claims, not on who holds them. Recall that  $II^N$  denotes the class of bijections from  $N$  into itself.

**Anonymity.** For each  $(c, E) \in \mathcal{C}^N$ , each  $\pi \in II^N$ , and each  $i \in N$ ,  $R_{\pi(i)}((c_{\pi(i)})_{i \in N}, E) = R_i(c, E)$ .

Another strengthening is the requirement that the rule should respect the ordering of claims: if agent  $i$ ’s claim is at least as large as agent  $j$ ’s claim, he should receive at least as much as agent  $j$  does; moreover, the differences, calculated agent by agent, between claims and awards, should be ordered as the claims are. This property appears in Aumann and Maschler (1985).<sup>31</sup>

**Order preservation.** For each  $(c, E) \in \mathcal{C}^N$  and each  $\{i, j\} \subseteq N$ , if  $c_i \geq c_j$ , then  $R_i(c, E) \geq R_j(c, E)$ , and  $c_i - R_i(c, E) \geq c_j - R_j(c, E)$ .

All of the rules of Section 3 satisfy this property but few satisfy *strict order preservation*, the requirement that if in addition, agent  $i$ ’s claim is greater than agent  $j$ ’s claim and  $E > 0$ , he should receive more (equality is not permitted any more), a parallel statement being made about losses.

*Group order preservation* goes beyond *order preservation* by applying the idea to groups: if the sum of the claims of the members of some group is at least as large as the sum of the claims of the members of some other group, then the sums of the awards to the members of the two groups should be related in a similar way (Thomson, 1998), a

<sup>30</sup>Lee (1994) discusses a weighted version of the constrained equal awards rule.

<sup>31</sup>Some authors only impose the first part.

parallel statement being made about losses. For each problem, the set of awards vectors satisfying these inequalities is a polygonal region; it is non-empty, as the proportional awards vector always belongs to it. Therefore, rules satisfying *group order preservation* are easily defined: for each problem, it suffices to select from the set. *Equal treatment of equal groups*, the counterpart for groups of *equal treatment of equals*, says that two groups with equal aggregate claims should receive equal aggregate awards.

Next is the requirement that claimants with greater claims should receive proportionately at most as much:

**Regressivity.** For each  $(c,E) \in \mathcal{C}^N$  and each  $\{i,j\} \subseteq N$ , if  $c_i \geq c_j > 0$ , then

$$\frac{R_i(c,E)}{c_i} \leq \frac{R_j(c,E)}{c_j}.$$

In the context of taxation, it is mainly the reverse inequality that has been imposed.

**Progressivity.** For each  $(c,E) \in \mathcal{C}^N$  and each  $\{i,j\} \subseteq N$ , if  $c_i \geq c_j > 0$ , then

$$\frac{R_i(c,E)}{c_i} \geq \frac{R_j(c,E)}{c_j}.$$

### 5.2. Monotonicity requirements

We turn to monotonicity requirements. Such requirements have played an important role in the analysis of other domains and they often have strong implications. Sometimes they are even incompatible with very elementary requirements of efficiency and fairness in distribution. In the present context, they are quite weak however.

First is the requirement that if an agent’s claim increases, he should receive at least as much as he did initially:<sup>32</sup>

**Claims monotonicity.** For each  $(c,E) \in \mathcal{C}^N$ , each  $i \in N$ , and each  $c'_i > c_i$ , we have  $R_i(c'_i, c_{-i}, E) \geq R_i(c, E)$ .

Under the same hypotheses, we might want each of the other claimants to receive at most as much as he did initially: For each  $(c,E) \in \mathcal{C}^N$ , each  $i \in N$ , and each  $c'_i > c_i$ , we have  $R_{N \setminus \{i\}}(c'_i, c_{-i}, E) \leq R_{N \setminus \{i\}}(c, E)$ . Together with *efficiency*, this requirement implies the previous one.<sup>33</sup> *Claims monotonicity*, and even this stronger version, are met very generally.

A requirement adapted from the international trade literature is that if an agent transfers part (or all) of his claim to the other agents he should receive at most as much as he did initially (Chun, 1988a). This requirement is easily met:

<sup>32</sup>By the notation  $c_{-i}$  we mean the vector  $c$  from which the  $i$ th coordinate has been removed.

<sup>33</sup>In the study of allocation in classical economies, requirements of this type, focusing on endowments, have played an important role (Thomson, 1987).

**No transfer paradox.** For each  $(c,E) \in \mathcal{C}^N$ , each  $i \in N$ , and each  $c' \in \mathbb{R}_+^N$ , if  $c'_i < c_i$  and  $\sum c'_j = \sum c_j$ , then  $R_i(c',E) \leq R_i(c,E)$ .

Alternatively, we could focus on a claim transfer from one agent to a specific other agent, and ask not only that the former should receive at most as much as he did initially, but also that the latter should receive at least as much as he did initially. This requirement is also easily satisfied.

The next requirement is that if the amount to divide increases, each claimant should receive at least as much as he did initially.<sup>34</sup>

**Resource monotonicity.** For each  $(c,E) \in \mathcal{C}^N$  and each  $E' \in \mathbb{R}_+$ , if  $\sum c_i \geq E' > E$ , then  $R(c,E') \geq R(c,E)$ .

Most of the rules that have been considered in the literature, and all of the rules that we have formally defined, are *resource monotonic*. However, the stricter requirement obtained by adding that under the same hypotheses, the inequalities appearing in the conclusion should be strict when possible (that is, if  $E > 0$  and for each claimant whose claim is positive) is not satisfied by most of them. A “conditional” version of it, resulting from imposing the strict inequality only for each claimant whose initial award is neither 0 nor equal to his claim (this eliminates “corner” situations), is satisfied more generally. (Similar comments apply to the conditions described earlier.)

The final requirement in this section is that if the amount to divide increases, of two agents, the one with the greater claim should receive a greater share of the increment than the other (Dagan et al., 1997).

**Super-modularity.** For each  $(c,E) \in \mathcal{C}^N$ , each  $E' \in \mathbb{R}_+$ , and each pair  $\{i,j\} \subseteq N$ , if  $\sum c_i \geq E' > E$  and  $c_i \leq c_j$ , then  $R_i(c,E') - R_i(c,E) \leq R_j(c,E') - R_j(c,E)$ .

Apart from the constrained egalitarian rule, all of the rules that we have seen satisfy this property. Here too, a strict version can be formulated, but it is rarely met. A *super-modular* rule satisfies the first part of *order preservation* but may violate the second part as well as *resource monotonicity*.

As their names indicate, both the constrained equal awards and constrained egalitarian rules are intended to achieve some objective of equality subject to constraints, and, in fact, the two rules can be characterized as best in that respect, differing only in the specification of these constraints. For the former, the only constraints are those imposed on rules (that they should select efficient vectors bounded below by 0 and above by the claims vector). For the latter, there is an additional constraint, that if the amount available is equal to the half-sum of the claims, each agent should receive his half-claim—let us refer to this property as the *midpoint property*—and *resource monotonicity*. The importance in the Talmud of the midpoint as a “psychological

<sup>34</sup>The property is considered by several authors, including Curiel et al. (1987) and Young (1988).

watershed” is documented by Aumann and Maschler (1985). Here are the characterizations:

**Theorem 3.** (Schummer and Thomson, 1997; Chun et al., 2001)

- (a) *The constrained equal awards rule is the only rule such that, for each problem, the gap between the smallest amount any claimant receives and the largest such amount is the smallest.*
- (b) *It is also the only rule such that, for each problem, the variance of the amounts received by all the claimants is the smallest.*
- (c) *The constrained egalitarian rule achieves gap minimization among all rules satisfying the midpoint property and resource monotonicity. A similar statement holds for variance minimization.*<sup>35</sup>

### 5.3. Independence, additivity, and related properties

In this section, we consider independence requirements of rules with respect to certain operations performed on the data of the problem.

First is the requirement that if claims and amount available are multiplied by the same positive number, then so should all awards. It is not always natural however to treat in a similar way situations in which claims and amount available are small, and situations in which these variables are large. For instance, in situations in which it is felt that each agent should be guaranteed a minimal amount if possible, this requirement is not reasonable. Similarly, in the context of taxation, one may want to exempt agents whose income is below some threshold.<sup>36</sup>

**Homogeneity.** For each  $(c, E) \in \mathcal{C}^N$  and each  $\lambda > 0$ ,  $R(\lambda c, \lambda E) = \lambda R(c, E)$ .

The next requirement is that the part of a claim that is above the amount to divide should be ignored. Since this part cannot be reimbursed anyway, replacing  $c_i$  by  $E$  for each  $i \in N$  such that  $c_i > E$  should not affect the chosen awards vector.<sup>37</sup>

**Invariance under claims truncation.** For each  $(c, E) \in \mathcal{C}^N$ ,  $R(c, E) = R((\min\{c_i, E\})_{i \in N}, E)$ .

<sup>35</sup>Results on Lorenz rankings of members of the ICI family, and of the CIC family, are provided by Thomson (2002), and parallel rankings for the subfamily introduced by Moreno-Ternero and Villar (2001) are offered by Moreno-Ternero and Villar (2002).

<sup>36</sup>However, as argued by Young (1988), the problem could then be redefined as pertaining to the division of whatever surplus exists after such thresholds have been reached.

<sup>37</sup>Note that the problem appearing in the property is well defined. The property is found in Curiel et al. (1987) and Dagan and Volij (1993). By analogy to the property studied in bargaining theory, these latter authors call it “independence of irrelevant claims”. We prefer the more neutral expression of “invariance under claims truncation” since it can be argued that the part of an agent’s claim that is above the amount available is not irrelevant.

This property is satisfied by the constrained equal awards, minimal overlap, random arrival, and Talmud rules, but not by the proportional or constrained equal losses rules. If we feel strongly that *invariance under claims truncation* should be imposed, we could of course redefine the domain and only consider problems in which no claim is greater than the amount to divide. Alternatively, we could limit attention to lists  $(c,E) \in [0,1]^N \times \mathbb{R}_+$ , where for each  $i \in N$ ,  $c_i$  is interpreted as the *percentage* of the amount to divide claimed by agent  $i$ . This restriction might be particularly meaningful in the context of estate division: think of contradictory wills each of which specifies the percentage of the estate that some heir should receive.<sup>38</sup>

Next, we require that the awards vector should be equivalently obtainable (i) directly, or (ii) by first assigning to each agent his minimal right, adjusting claims down by these amounts, and finally, applying the rule to divide the remainder.

**Minimal rights first.** For each  $(c,E) \in \mathcal{C}^N$ ,

$$R(c,E) = m(c,E) + R(c - m(c,E), E - \sum m_i(c,E)).^{39}$$

This property is satisfied by the constrained equal losses, random arrival, and Talmud rules, but not by the proportional, constrained equal awards, or minimal overlap rules (these facts are discussed by Thomson and Yeh, 2001).

Now, consider the following situation: after having divided the liquidation value of a firm among its creditors, its assets are re-evaluated and found to be worth less than initially thought (perhaps their market value has changed in the meantime, or certain assets are found to be inaccessible). To deal with the new situation, two options are available: (i) we cancel the initial division and apply the rule to the revised problem; or (ii) we consider the initial awards as claims on the revised value and apply the rule to the problem so defined. Our next requirement is that both ways of proceeding should result in the same awards vectors (Moulin, 2000).

**Composition down.** For each  $(c,E) \in \mathcal{C}^N$  and each  $E' < E$ , we have  $R(c,E') = R(R(c,E), E')$ .

*Composition down* is satisfied by the proportional, constrained equal awards, and constrained equal losses rules but not by the random arrival, Talmud, or minimal overlap rules.

The opposite possibility is just as plausible, namely that after having divided the liquidation value of a bankrupt firm among its creditors, its assets are re-evaluated but this time they are found to be worth more than originally thought. We have two parallel options: (i) we cancel the initial division and apply the rule to the revised problem, or (ii) we let agents keep their initial awards, adjust claims down by these amounts, and

<sup>38</sup>O'Neill (1982) calls these problems "simple claims problems".

<sup>39</sup>This problem is well defined since for each  $(c,E) \in \mathcal{C}^N$ ,  $c \geq m(c,E)$ ,  $E - \sum m_i(c,E) \geq 0$ , and  $\sum (c_i - m_i(c,E)) \geq E - \sum m_i(c,E)$ . The property is introduced by Curiel et al. (1987) under the name of the "minimal rights property". Dagan (1996) refers to it as "v-separability".

reapply the rule to divide the incremental worth. The requirement formulated next is that both ways of proceeding should result in the same awards vectors (Young, 1988).<sup>40</sup>

**Composition up.** For each  $(c,E) \in \mathcal{C}^N$  and each  $E' \in \mathbb{R}_+$ , if  $\sum c_i \geq E' > E$ , then  $R(c,E') = R(c,E) + R(c - R(c,E), E' - E)$ .<sup>41</sup>

*Composition up* bears a resemblance to *minimal rights first* but there is no logical relation between the two: the constrained equal awards rule satisfies *composition up* but not *minimal rights first*, whereas the opposite holds for the random arrival rule. Also, *composition up* is satisfied or violated by the same examples we gave for *composition down*, but the two composition properties are not logically related either.

The next requirement is that no group of agents should receive more by transferring claims among themselves (Chun, 1988a):<sup>42</sup>

**No advantageous transfer.** For each  $(c,E) \in \mathcal{C}^N$ , each  $M \subset N$ , and each  $(c'_i)_{i \in M} \in \mathbb{R}_+^M$ , if  $\sum_M c_i = \sum_M c'_i$ , then  $\sum_M R_i(c,E) = \sum_M R_i((c'_i)_{i \in M}, c_{N \setminus M}, E)$ .<sup>43</sup>

Obviously, in the presence of *efficiency* (which is incorporated in our definition of a rule), this property is vacuously satisfied for  $|N|=2$ . Of our main rules, only the proportional rule passes this test.

We continue with a somewhat more technical requirement: the awards vector should be a linear function of the amount to divide:

**Resource linearity.** For each  $(c,E) \in \mathcal{C}^N$ , each  $E'$  such that  $\sum c_i \geq E'$ , and each  $\lambda \in [0,1]$ , we have  $R(c, \lambda E + (1 - \lambda)E') = \lambda R(c,E) + (1 - \lambda)R(c,E')$ .

The proportional rule is the only one of our main rules to satisfy the property.

The next requirement pertains to situations in which the amount to divide comes in two parts. It states that dividing the first part first and then dividing the second part, no adjustment in claims being made, should yield the same awards vector as consolidating the two parts and dividing the sum at once. The implications of this property are studied by Chun (1988a). It is most appealing in situations in which the vector  $c$  is given a broader interpretation than a vector of claims as understood so far, but instead represents rights that are not commensurable with what is to be divided.<sup>44</sup> Then, the fact that a first

<sup>40</sup>A property of *step by step negotiation* in the same spirit is analyzed in the context of bargaining by Kalai (1977).

<sup>41</sup>The problem appearing in this expression is a well-defined problem since rules satisfy *non-negativity* and *claims boundedness*.

<sup>42</sup>A property of this type is considered by Gale (1974) and Aumann and Peleg (1974) in the context of allocation in classical exchange economies, and by Moulin (1985a,b) in the context of quasi-linear social choice.

<sup>43</sup>By the notation  $((c'_i)_{i \in M}, c_{N \setminus M})$ , we mean the vector in which the claim of each  $i \in M$  is  $c'_i$ , and the claim of each  $i \in N \setminus M$  is  $c_i$ .

<sup>44</sup>Then of course the inequality  $\sum c_i \geq E$  has no meaning and it makes sense to enlarge the class of problems under consideration by dropping it.

amount has already been allocated cannot be very meaningfully accompanied by an adjustment in claims when the second amount becomes available.

**Resource additivity.** For each  $(c, E) \in \mathcal{C}^N$  and each pair  $\{E', E''\}$  of elements of  $\mathbb{R}_+$ , if  $E = E' + E''$ , then  $R(c, E) = R(c, E') + R(c, E'')$ .

The proportional rule satisfies the property. A stronger version is obtained by dropping the hypothesis of equal claims vector  $s$ . No rule satisfies it, but if *non-negativity* is dropped, then it can be met.

The next requirement is sometimes needed for technical reasons, but it makes much intuitive sense, and it is satisfied by all of the rules that have been considered in the literature: simply, small changes in the data of the problem should not lead to large changes in the chosen awards vector.

**Continuity.** For each sequence  $\{(c^v, E^v)\}$  of elements of  $\mathcal{C}^N$  and each  $(c, E) \in \mathcal{C}^N$ , if  $(c^v, E^v)$  converges to  $(c, E)$ , then  $R(c^v, E^v)$  converges to  $R(c, E)$ .

Partial notions of continuity, with respect to the amount to divide, with respect to each agent's claim separately, or with respect to the entire claims vector, can also be formulated.

Our last requirement for the fixed-population version of the model is that the problem of dividing “what is available” and the problem of dividing “what is missing” should be treated symmetrically. The property is formulated by Aumann and Maschler (1985), who note several passages in the Talmud where the idea is implicit:

**Self-duality.** For each  $(c, E) \in \mathcal{C}^N$ ,  $R(c, E) = c - R(c, \sum c_i - E)$ .<sup>45</sup>

Many rules are *self-dual*, including the proportional, Talmud, and random arrival rules.<sup>46</sup>

An operation associating to each rule its “dual” can be defined as follows: Given  $R$ , the *dual of  $R$*  selects, for each problem  $(c, E) \in \mathcal{C}^N$ , the awards vector  $c - R(c, \sum c_i - E)$  (the right-hand side of the formula appearing in the statement of *self-duality*). It is clear that the constrained equal awards and constrained equal losses rules are dual. To say that a rule is *self-dual* is to say that it coincides with its dual.

Now, given two properties, we say that they are *dual of each other* if whenever a rule satisfies one of them, its dual satisfies the other. *Composition down* and *composition up* are dual (Moulin, 2000). So are *invariance under claims truncation* and *minimal rights*

<sup>45</sup>The problem appearing in this property is well defined since rules satisfy *claims boundedness*.

<sup>46</sup>Aumann and Maschler (1985) observe that the Talmud rule is the only *self-dual* rule that coincides with the constrained equal awards rule on the subdomain of problems  $(c, E)$  such that  $\sum (c_i/2) \geq E$ , provided the half-claims are used instead of the claims themselves (Aumann and Maschler, 1985). More generally, consider a rule defined for each claims vector and for each amount to divide running from 0 to the half-sum of the claims. If for the half-sum of the claims, it selects the half-claims vector, then it has a unique *self-dual* extension to all admissible values of the resource.

first (Herrero and Villar, 2001a; Dagan, 1996, proves a related result). *Conditional full compensation* and *conditional null compensation* are too (Herrero and Villar, 2001b).

A property is *self-dual* if whenever a rule satisfies the property, so does its dual. Examples of *self-dual* properties are *equal treatment of equals*, *resource monotonicity*, and *continuity*. The duality notion leads us to the formulation of new properties. Of particular interest is the *dual of claims monotonicity*. It says that if an agent's claim and the amount to divide increase by the same amount, the agent's award should increase by at most that amount (Thomson and Yeh, 2001).

These duality notions also allow us to derive from each characterization of a rule a characterization of the dual rule by simply replacing each property by its dual (Herrero and Villar, 2001a, and Moulin, 2000, exploit this fact). The same comment applies to a characterization of a family of rules. Here, the dual result is a characterization of the dual family. The theorems stated below include a number of such pairs of dual characterizations. Mostly, these characterizations are grouped according to which rule comes out of the axioms.

**Theorem 4.** *The constrained equal awards rule is the only rule satisfying*

- (a) *equal treatment of equals, invariance under claims truncation, and composition up (Dagan, 1996);*
- (b) *conditional full compensation and composition down (Herrero and Villar, 2001b);*<sup>47</sup>
- (c) *conditional full compensation and claims monotonicity (Yeh, 2001a).*

Dual characterizations of the constrained equal losses rules follow:

**Theorem 5.** *The constrained equal losses rule is the only rule satisfying*

- (a) *equal treatment of equals, minimal rights first, and composition down (Herrero, 2001);*
- (b) *conditional null compensation and composition up (Herrero and Villar, 2001b);*
- (c) *conditional null compensation and the dual of claims monotonicity (Yeh, 2001a).*

The next two theorems pertain to the two-claimant case:

**Theorem 6.** (Dagan, 1996) *For  $|N| = 2$ . Concede-and-divide is the only rule satisfying*

- (a) *invariance under claims truncation and self-duality;*
- (b) *minimal rights first and self-duality;*
- (c) *equal treatment of equals, invariance under claims truncation, and minimal rights first.*

Our next characterization, which also pertains to the two-claimant case, identifies a

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<sup>47</sup>A simple proof is given by Yeh (2001b).

family of two-claimant rules that “connect” the proportional, constrained equal awards, and constrained equal losses rules (Moulin, 2000).

**Family  $\mathcal{D}$ .** For  $|N| = 2$ . Each member of the family is defined as follows. Awards space is partitioned into cones; each non-degenerate cone is spanned by a homothetic family of piece-wise linear curves in two pieces, a segment containing the origin and contained in one of the boundary rays of the cone (the “first ray”) and a half-line parallel to the other boundary ray (the “second ray”). (Cones can be degenerate, that is, can be rays.) For each claims vector, the path of awards of the rule is obtained by identifying the cone to which the claims vector belongs and the curve in the cone passing through it. The path is the restriction of the curve to the box having the origin and the claims vector as opposite vertices and whose sides are parallel to the axes.

The family  $\mathcal{D}$  is large—note in particular that many of its members violate our symmetry requirements. To obtain *equal treatment of equals*, requires the 45° line to be a cone in the partition. For *anonymity*, require the partition to be symmetric with respect to the 45° line, and the designation of rays as first or second in each pair of symmetric cones to share this symmetry.

**Theorem 7.** (Moulin, 2000) *For  $|N| = 2$ . The members of the family  $\mathcal{D}$  are the only rules satisfying homogeneity, composition down, and composition up.*

Returning to the case of an arbitrary number of claimants, we offer several characterizations of the proportional rule.<sup>48</sup>

**Theorem 8.** *The proportional rule is the only rule satisfying*

- (a) *self-duality and composition up* (Young, 1988);
- (b) *self-duality and composition down*;
- (c) *for  $|N| \geq 3$ ; no advantageous transfer* (Moulin, 1985a,b; Chun, 1988a; Ju and Miyagawa, 2002);
- (d) *resource linearity* (Chun, 1988a).

#### 5.4. Operators

Several properties defined in the previous sections suggest the definition of operators on the space of rules. Let  $R$  be a rule. The *claims truncation* operator associates with  $R$  the rule  $R^t$  defined as follows: for each problem, first truncate claims and then apply  $R$ :  $R^t(c, E) = R(\min\{c_i, E\}_{i \in N}, E)$ . The *attribution of minimal rights* operator associates with  $R$  the rule  $R^m$  defined as follows: for each problem, first assign to each claimant his minimal right, revise claims down by these amounts, and the amount to divide by their

<sup>48</sup>Part (c) involves neither *continuity* nor *equal treatment of equals*. These properties were included in early versions of the uniqueness part. Moulin’s (1985a,b) result pertains to a related class of problems.

sum; now, apply  $R$  to the resulting problem, thereby obtaining a second round of payments:

$$R^m(c,E) = m(c,E) + R(c - m(c,E),E - \sum m_i(c,E)).$$

We have already defined the *duality* operator. Finally, given a list of rules and a list of non-negative weights for them, the *convexity* operator produces the rule that associates with each problem the weighted average of the awards vectors chosen by these rules for the problem.

A systematic analysis of these operators is carried out by Thomson and Yeh (2001), who establish the following structural relations between them.

**Theorem 9.** (Thomson and Yeh, 2001)

- (a) *If two rules are dual, the version of one obtained by subjecting it to the claims truncation operator and the version of the other obtained by subjecting it to the attribution of minimal rights operators are dual.*
- (b1) *The claims truncation and attribution of minimal rights operators commute: starting from any rule, subjecting it to these two operators, in either order, produces the same rule.*
- (b2) *In the two-claimant case, starting from any rule satisfying equal treatment of equals, subjecting it to the two operators produces concede-and-divide (by (b1), the order does not matter).*

When a rule satisfies a property of interest, a natural question is whether the rule obtained by subjecting it to a certain operator still does. The lists of which properties are preserved by the four operators and which are not are drawn by Thomson and Yeh (2001). The duality operator preserves many properties, an exception being *claims monotonicity*. The convexity operator also preserves many properties. Here, exceptions are *composition down* and *composition up*. The claims truncation and attribution of minimal rights operators are more disruptive. Obviously, they preserve neither *self-duality* nor *no advantageous transfer*, but they also fail to preserve *composition down*, *composition up*, and perhaps more surprisingly, *resource monotonicity*. They are comparable in this respect, an implication of the following theorem:

**Theorem 10.** *A property is preserved under the claims truncation operator if and only if its dual is preserved under the attribution of minimal right operator.*

## 6. Properties of rules: the variable-population case

We now consider a framework in which the population of claimants involved may vary. We consider all problems with an arbitrary, although finite, number of them drawn from that population. Formally, there is a set of “potential” claimants, indexed by the natural numbers  $\mathbb{N}$ . Let  $\mathcal{N}$  be the class of non-empty finite subsets of  $\mathbb{N}$ . A *claims*

*problem* is defined by first specifying  $N \in \mathcal{N}$ , then a pair  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$  such that  $\sum_N c_i \geq E$ . We still denote the class of all problems with claimant set  $N$  by  $\mathcal{C}^N$ , so that a *rule* is a function defined on the union of all of the  $\mathcal{C}^N$ 's, when  $N$  ranges over  $\mathcal{N}$ , which associates with each  $N \in \mathcal{N}$  and each  $(c, E) \in \mathcal{C}^N$  an awards vector for  $(c, E)$ .

In all of the results presented in this section, the axioms introduced earlier in the fixed-population case are generalized in the obvious way to be applicable to variable populations. Rules are generalized in a similar way. For instance, when we refer to the “proportional rule”, we now mean the rule defined on  $\cup_{N \in \mathcal{N}} \mathcal{C}^N$  obtained by applying the proportional formula for each population of claimants and each problem they may face.

### 6.1. Population monotonicity

The monotonicity property that is relevant in the context of a variable population is that if the population of claimants enlarges but the amount to divide stays the same, each of the claimants initially present should receive at most as much as he did initially.<sup>49</sup>

**Population monotonicity.** For each pair  $\{N, N'\} \subset \mathcal{N}$  such that  $N' \supset N$ , and each  $(c, E) \in \mathcal{C}^N$ ,  $R_N(c, E) \leq R(c_{N'}, E)$ .

Like all of the fixed-population monotonicity properties, this property is rather weak: all of the rules that have been studied in the literature satisfy it, although here too, the stricter requirement obtained by adding that if  $E > 0$ , the difference between the final and initial awards to any claimant initially present whose claim is positive should also be positive, is considerably more restrictive. A conditional version of it, obtained by applying it only to the agents whose initial awards are neither 0 nor equal to their claims, is met much more generally.

### 6.2. Replication invariance

Another requirement pertaining to population changes is *replication invariance*: if a problem is replicated  $k$  times, the awards vector that is chosen should be the  $k$ -times replica of the awards vector chosen for the problem subjected to the replication. Replicating economies is a standard way in economic theory of gaining insight into situations where all agents are “small” and have a negligible impact on equilibrium variables.

Many rules are *replication invariant*. Examples that fail the test are the random arrival and minimal overlap rules. However, for them, we have the following limit result, which essentially says that under sufficient replication, they “behave” like the proportional and constrained equal losses rules respectively:

<sup>49</sup>The property is introduced, in the context of bargaining, by Thomson (1983). For a survey of the literature on *population monotonicity*, see Thomson (1995).

**Theorem 11.** (Chun and Thomson, 2000)

- (a) *The random arrival awards vector of a replicated problem is the replica of an awards vector of the problem subjected to the replication that, as the order of replication increases without bound, converges to its proportional awards vector.*
- (b) *For the minimal overlap rule, a similar result holds with the limit awards vector being the constrained equal losses awards vector.*

As we noted, it is easy to construct rules satisfying *equal treatment of equal groups*, but when combined with *replication invariance* and the very mild axiom of *claims continuity*—this requirement is met by all standard rules—only the proportional rule remains admissible:

**Theorem 12.** (Ching and Kakkar, 2000; Chambers and Thomson, 2002) *The proportional rule is the only rule satisfying equal treatment of equal groups, claims continuity, and replication invariance.*

If *equal treatment of equal groups* is strengthened to *group order preservation* in addition to *replication invariance*, another characterization of the proportional rule obtains, without any continuity property being imposed (Chambers and Thomson, 2002).

### 6.3. Consistency

Our next requirement expresses a certain kind of independence of rules with respect to variations in populations. Given a rule, consider some problem, apply the rule to it, and imagine some claimants leaving with their awards. The requirement is that when the situation is re-evaluated from the viewpoint of the remaining claimants, the rule should award to each of them the same amount as it did initially. The problem faced by the remaining claimants is defined in the following natural way: their claims are unchanged and the amount to divide is the difference between the amount initially available and the sum of the awards to the agents who left (alternatively the sum of the amounts intended for them, the agents who stay). It is called the *reduced problem relative to the subgroup and the initial recommendation*. Note that since, by definition, rules satisfy *claims boundedness*, the sum of the claims of the remaining claimants is still at least as large as the amount left for them, so that the reduction produces a well-defined claims problem.<sup>50</sup>

**Consistency.** For each pair  $\{N, N'\}$  of elements of  $\mathcal{N}$  such that  $N' \subset N$ , and each  $(c, E) \in \mathcal{C}^N$ , if  $x \equiv R(c, E)$ , then  $x_{N'} = R(c_{N'}, \Sigma_{N'} x_i)$ .

<sup>50</sup>For a survey of the vast literature devoted to the analysis of the consistency principle, see Thomson (1996). O'Neill (1982) gives the term “consistency” a different meaning.

*Bilateral consistency* is the weaker property obtained by considering only subgroups of two remaining agents (add the proviso  $|N'| = 2$ ).

The next requirement pertains to situations in which some agents' claims are equal to 0: by definition of a rule, these agents get nothing; the requirement is that deleting them should leave the amounts received by the others unaffected.<sup>51</sup> The property is a weak form of *consistency*. It appears in O'Neill (1982) and Chun (1988a).<sup>52</sup>

**Limited consistency.** For each pair  $\{N, N'\}$  of elements of  $\mathcal{N}$  such that  $N' \subset N$ , and each  $(c, E) \in \mathcal{C}^N$ , if for each  $i \in N \setminus N'$ ,  $c_i = 0$ , then  $R_{N'}(c, E) = R(c_{N'}, E)$ .

*Converse consistency* says that, given some problem and some awards vector for the problem, if this vector is such that for each two-claimant subgroup of claimants, its restriction to the subgroup would be chosen by the rule for the reduced problem associated with it, then it should be chosen for the initial problem.

Any rule satisfying *equal treatment of equals* and *consistency* is *replication invariant*. Logical relations between *consistency*, its *converse*, and various fixed-population properties are described in detail by Chun (1999). It is also of interest that it sometimes happens that a property, when imposed for the two-claimant case on a rule that is required to be *consistent*, automatically holds for more than two claimants. Many properties are so “lifted” but certain important ones are not, *continuity* being an example. On occasion, lifting occurs if the rule satisfies some other basic property. *Resource monotonicity*, which itself is lifted, is very helpful in lifting other properties. Lists of properties that are lifted, alone or with the help of auxiliary properties, are drawn by Hokari and Thomson (2001).

It is clear that the proportional rule is *consistent*, and that so are the constrained equal awards and constrained equal losses rules. On the other hand, none of the following rules is *consistent*: the adjusted proportional, random arrival, and minimal overlap rules. What of the Talmud rule? Let us check with the specific numerical values given in the Talmud (Fig. 1b), and starting with one of the three-claimant estate division problems. For an estate of 200, the awards to claimants 1 and 2 are 50 and 75, respectively, for a total of 125. Applying the contested garment rule to divide 125 between them returns the same numbers, 50 and 75. In fact, given any value of the estate, if  $x$  denotes the Talmud awards vector for the three-claimant problem, then, given any pair of claimants  $\{i, j\}$ , applying the contested garment rule to divide  $x_i + x_j$  between them yields the awards vector  $(x_i, x_j)$ . This coincidence occurs generally and, thus, the Talmud rule is *consistent*. The following result provides a justification for the rule on the basis of *bilateral*

<sup>51</sup>The requirement that any agent whose claim is 0 should receive nothing corresponds to the property known in the theory of coalitional games as the “dummy property”. It appears explicitly in some results pertaining to generalized rules (Chun, 1988a; de Frutos, 1999). It is automatically satisfied by rules as we defined them.

<sup>52</sup>Chun uses the term “dummy” for the conjunction of what we call *dummy* and *limited consistency*.

*consistency* and the choice of concede-and-divide for the two-claimant case, a choice that we have seen can be justified from a wide range of perspectives.<sup>53</sup>

**Theorem 13.** (Aumann and Maschler, 1985) *The Talmud rule is the only bilaterally consistent rule to coincide with concede-and-divide in the two-claimant case.*

An interesting connection exists between the reduction operation in the space of claims problems and a counterpart of this operation in the space of TU coalitional games. Let  $N \in \mathcal{N}$ ,  $(c, E) \in \mathcal{C}^N$ , and  $x$  be an awards vector for  $(c, E)$ . Given  $N' \subset N$ , consider the reduced problem  $(c_{N'}, \sum_{N'} x_i)$  and its associated coalitional game  $v(c_{N'}, \sum_{N'} x_i)$  (Section 4.2). Also, calculate the coalitional game associated with the problem  $(c, E)$ ,  $v(c, E)$ , and then its “reduced game with respect to  $N'$  and  $x$ ” (Davis and Maschler, 1965): in this game, the worth of each coalition  $S$  is defined to be the maximal surplus obtained by the coalition when it “cooperates” with a subset  $\bar{S}$  of the complementary group  $N \setminus N'$ —this yields  $v(c, E)(S \cup \bar{S})$ —and pays the members of  $\bar{S}$  according to  $x$ ; the surplus is the difference  $v(c, E)(S \cup \bar{S}) - \sum_{\bar{S}} x_i$ . The maximization is carried over all  $\bar{S} \subset N \setminus N'$ . Remarkably, the two ways of proceeding give the same game (Aumann and Maschler, 1985).<sup>54</sup>

The implications of *consistency* have been described very completely, with very few auxiliary properties. First, it is straightforward to check that all parametric rules are *consistent*.<sup>55</sup> Fig. 4 depicts, for a parametric rule of parameterization  $f$ , the graphs of  $f(c, \cdot)$  for three possible values of the first argument, called  $c_1$ ,  $c_2$ , and  $c_3$ . The choice of  $\lambda$  produces the distribution (50,60,100), and the choice of  $\lambda'$  the distribution (55,80,120). Note that two of the graphs are not strictly increasing and that the graph corresponding to  $c_2$  does not lie entirely above that corresponding to  $c_1$  even though  $c_1 < c_2$ . At this stage, these are indeed possibilities. It is clear however that they are eliminated by imposing additional requirements on rules. For instance, for an *order preserving* rule of representation  $f$ , the graph of  $f(\bar{c}, \cdot)$  lies everywhere on or above the

<sup>53</sup>It follows directly from the fact that the Talmud rule is *conversely consistent*, using the Elevator Lemma (Thomson, 1996): if a *bilaterally consistent* rule coincides with a *conversely consistent* rule in the two-claimant case, coincidence holds in general. Actually, they establish a slightly stronger result, namely, that even if the rule were allowed to be multivalued, then *bilateral consistency* and coincidence with the contested garment rule in the two-claimant case would imply (i) *singlevaluedness*, and (ii) coincidence with the Talmud rule (and therefore uniqueness). A similar application of the Elevator Lemma tells us that the constrained equal awards rule is the only one to satisfy *exemption*, *composition down* and *bilateral consistency* (Herrero and Villar, 2001a,b; Yeh, 2001b). This characterization exploits Theorem 46b and the fact that, in the two-claimant case, *conditional full compensation* is the same thing as *exemption*. A characterization of the constrained equal losses rule is obtained by duality: it is the only rule to satisfy *exclusion*, *composition up* and *bilateral consistency*.

<sup>54</sup>Lee (1994) develops a characterization of the weighted constrained equal awards rules based on *consistency*. This characterization exploits certain duality relations between cores, anticores and their reductions.

<sup>55</sup>Note that, if in the definitions of these rules, we choose a different function  $f$  for each agent, we preserve *consistency*, but not *equal treatment of equals*.

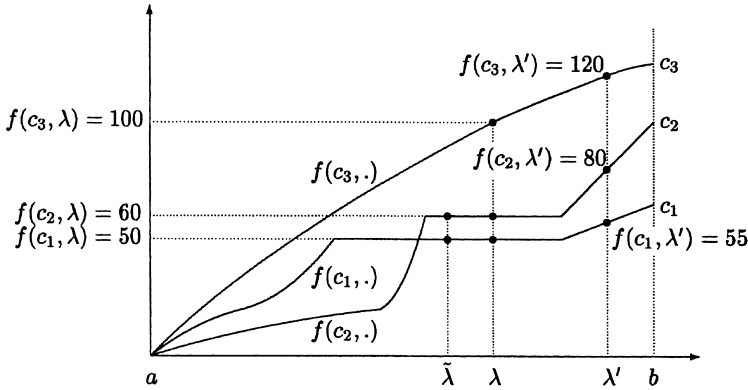


Fig. 4. Parametric rules are consistent. To solve the problem  $(c_1, c_2, c_3; E)$ , we first identify a value  $\tilde{\lambda} \in [a, b]$  such that  $f(c_1, \tilde{\lambda}) + f(c_2, \tilde{\lambda}) + f(c_3, \tilde{\lambda}) = E$  ( $\tilde{\lambda} = \lambda$  in the figure if  $E = 210$ ). These are the awards to the three claimants. Then, we imagine the departure of the third claimant with his award. Now, if the amount  $E' \equiv E - f(c_3, \lambda)$  (in the figure,  $E' = 110$ ) is to be divided between claimants 1 and 2, the same value  $\lambda$  of the parameter guarantees that the awards  $f(c_1, \lambda)$  and  $f(c_2, \lambda)$  add up to  $E'$ , so that claimants 1 and 2 still receive the same amounts after claimant 3 has received  $f(c_3, \lambda)$ . There could be other values of the parameters solving the required equation (in the figure,  $\tilde{\lambda}$  will work as well), but they would result in the same awards vector.

graph of  $f(\bar{c}, \cdot)$  whenever  $\bar{c} > c$ . Also, for a *super-modular* rule, for each value of the parameter  $\lambda$ , the slope of the graph of  $f(\bar{c}, \cdot)$  is everywhere at least as large as the slope of the graph of  $f(c, \cdot)$  whenever  $\bar{c} > c$  (if these slopes are well defined). One of the most important results in the theory under review is the following characterization of the parametric family (Young, 1987a).

**Theorem 14.** (Young, 1987a<sup>56</sup>) *The parametric rules are the only rules satisfying continuity, equal treatment of equals, and bilateral consistency.*

It is of interest that the proof includes showing that a rule satisfying *equal treatment of equals, continuity and consistency*, is *resource monotonic*.

Also, if a rule  $R$  satisfies the properties of Theorem 14, there is a function  $H : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  such that for each  $c_0 \in \mathbb{R}_+$ ,  $H(c_0, \cdot)$  is continuous, increasing, and strictly convex with respect to its second argument, and for each  $N \in \mathcal{N}$  and each  $(c, E) \in \mathcal{C}^N$  with  $c > 0$ ,  $R(c, E)$  is obtained by minimizing the expression  $\sum_N H(c_i, x_i)$  over  $x \in \mathbb{R}_+^N$  satisfying  $\sum_N x_i = E$  and  $x \leq c$  (Young, 1987a).

Consider now the following family of rules: Let  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a continuous and strictly increasing function such that  $\lim_{x \rightarrow 0^+} u(x) = -\infty$ . Then, for each  $N \in \mathcal{N}$  and each  $(c, E) \in \mathcal{C}^N$  with  $c > 0$ , the *equal-sacrifice rule relative to  $u$*  selects the awards vector  $x$  for  $(c, E)$  such that for some  $\lambda \geq 0$ , and for each  $i \in N$ , we have  $u(c_i) - u(c_i - x_i) = \lambda$ .

<sup>56</sup>Kaminski (2000, 2001) provides further detail.

**Theorem 15.** (Young, 1988) *On the domain of problems with positive claims, the equal-sacrifice rules are the only rules satisfying continuity, equal treatment of equals, strict resource monotonicity, strict order preservation for losses,<sup>57</sup> composition up, and consistency. If, in addition, homogeneity is imposed, then the rule is an equal-sacrifice rule relative to a function  $u$  such that either  $u(x) \equiv \ln(x)$  or  $u(x) \equiv -x^p$  for  $p < 0$ .*

In the context of taxation, the case described first in the second part of the theorem corresponds to proportional taxation (also called “flat taxation”), and the second case to a parametric rule for which, for each  $i \in N$ ,  $f(c_i, \lambda) \equiv c_i - [c_i^p + \lambda^p]^{1/p}$  for  $\lambda \in [0, \infty)$ , where  $p < 0$ . (For further studies of the issue, see Young, 1987b, 1990.)

We saw in Section 6.2 that the proportional rule is the only rule satisfying *group order preservation* and *replication invariance*. Suppose now that the population of potential claimants is finite but has at least three members. Then, the rule is the only one to satisfy *claims continuity*, *equal treatment of equal groups*, and *consistency* (Chambers and Thomson, 2002).

Next, we address the issue of extending two-claimant rules to general populations so as to obtain *consistency*. *Consistency* implies that the path of awards for an arbitrary claims vector, when projected onto the subspace pertaining to any two-claimant subgroup of the claimants it involves, is a subset of the path of awards for the projection of the claims vector onto that subspace, and coincides with this path if in addition the rule is *resource monotonic*. This simple observation underlies a technique to determine whether a rule specified for the two-claimant case has a *consistent* extension, and if it does, how to construct it (Thomson, 2001a). The technique is particularly useful for rules whose paths of awards are piece-wise linear, a frequent occurrence. We give three applications.

The first one concerns the existence of weighted versions of the Talmud rule. Recall that concede-and-divide is the only rule defined on  $\mathcal{C}^N$ , where  $|N| = 2$ , satisfying *equal treatment of equals*, *invariance under claims truncation*, and *minimal rights first* (Theorem 6c). If *equal treatment of equals* is dropped and *homogeneity* is added, we obtain a one-parameter family, indexed by  $\alpha \in \Delta^N$ , and defined as follows: for each  $i \in N$ , and each  $(c, E) \in \mathcal{C}^N$ , the *weighted concede-and-divide rule relative to the weights  $\alpha$*  assigns each claimant  $i \in N$  the amount

$$CD_i^\alpha(c, E) \equiv \max\{E - c_j, 0\} + \alpha_i[E - \max\{E - c_j, 0\} - \max\{E - c_i, 0\}].$$

We will ask how to extend these rules to general populations in a *consistent* manner. The second application addresses the question of the existence of *consistent* extensions of the members of the family  $\mathcal{D}$  of two-claimant rules characterized in Theorem 7. The third one answers a similar question about the average of the constrained equal awards and constrained equal losses rules. For the first two applications, we need to define two additional families of rules:

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<sup>57</sup>The second part of what we called *order preservation*.

**Family  $\mathcal{T}$ .** Each member of the family is defined as follows. The population of potential claimants is partitioned into priority classes, and for each two-claimant class, a weight vector is specified. To solve each problem, we first identify the partition of the set of claimants actually present induced by the reference partition. The elements of this partition are handled in succession. For each class induced from a two-claimant reference class, the weighted concede-and-divide rule relative to the weights for that class is applied; for each other class, the Talmud rule is applied.

**Family  $\mathcal{M}$ .** Each member of the family is defined as follows. The population of potential claimants is partitioned into priority classes; for each two-claimant class, a rule in the family  $\mathcal{D}$  is specified; to each class with three or more claimants, one of the following labels is attached: “proportional”, or “constrained equal awards”, or “constrained equal losses”, and in each of the last two cases, a positive weight is specified for each claimant in the class. To solve each problem, we first identify the partition of the set of claimants actually present induced by the reference partition. The elements of this partition are handled in succession. For each class induced from a two-claimant reference class, the rule in  $\mathcal{D}$  specified for that class is applied; for each class induced from a three-or-more claimants reference class, the proportional, or weighted constrained equal awards, or weighted constrained equal losses rule is applied, according to the label attached to the class, with weights proportional to the weights that have been assigned to the agents who are present.

**Theorem 16.** (a) *A rule satisfies homogeneity, invariance under claims truncation, minimal rights first, and consistency if and only if it belongs to the family  $\mathcal{T}$  (Hokari and Thomson, 2003).*

(b) *A rule satisfies homogeneity, composition down, composition up, and consistency if and only if it belongs to the family  $\mathcal{M}$  (Moulin, 2000; for an alternative proof, see Thomson, 2001b).*

(c) *A rule coincides with a weighted average of the constrained equal awards and constrained equal losses rule in the two-claimant case and satisfied consistency if and only if in fact all the weight is placed on one of these two rules, or all the weight is placed on the other (Thomson, 2001a).*

We conclude this discussion of *consistency* with an additional justification for the Talmud rule (Aumann and Maschler, 1985). Let  $N \equiv \{1, \dots, n\}$  and suppose that claimants are ordered by increasing claims. First, apply the contested garment rule to the two-claimant problem in which the first claimant faces a “composite claimant” whose claim is the sum  $c_2 + \dots + c_n$ . The first claimant leaves with his award unless a violation of *order preservation* occurs, in which case equal division takes place and we are done. Otherwise the second claimant faces a second composite claimant whose claim is the sum  $c_3 + \dots + c_n$  and the amount to divide is what the first composite claimant received. Claimant 2 leaves with his award unless a violation of *order preservation* occurs, in which case equal division of what was left takes place among the members of

$N \setminus \{1\}$  and we are done. The process continues in this way for at most  $n - 1$  steps. It leads to the Talmud awards vector.<sup>58</sup>

#### 6.4. Average consistency

Consider a rule that is not *consistent*. Then, there is at least one problem—let the recommendation made by the rule for it be denoted  $x$ —at least one subgroup of claimants and at least one claimant in the subgroup, say claimant  $i$ , such that in the reduced problem associated with the subgroup and  $x$ , he receives an amount that is different from what he was initially awarded,  $x_i$ . Because of *efficiency*, in the reduced problem, at least one claimant receives less, and at least one other claimant receives more, than initially decided. Of course, a claimant receiving less in some reduced problem associated with  $x$  may receive more in some other reduced problem associated with  $x$ . Suppose however that for each claimant, on average, when all the reduced problems associated with  $x$  relative to subgroups to which he belongs are considered, he does receive his component of  $x$ . Then, we may be satisfied with  $x$  after all. To the extent that the formation of subgroups is a thought experiment anyway, this weaker notion may be quite acceptable.

**Average consistency.** For each  $N \in \mathcal{N}$ , each  $(c, E) \in \mathcal{C}^N$ , and each  $i \in N$ ,

$$x_i = \frac{1}{(|N| - 1)!} \sum_{M \subset N, i \in M} R_i(c_M, \sum_M x_j).$$

This form of *consistency* is studied by Dagan and Volij (1997) who suggest that the averaging be limited to groups of size two. We refer to this version as *2-average consistency*.<sup>59</sup> They have in mind situations in which a rule for the two-claimant case has been chosen. Then, the idea of *2-average consistency* can be exploited to provide an extension of the rule to any number of claimants as follows: given  $N \in \mathcal{N}$  and  $(c, E) \in \mathcal{C}^N$ , select  $x \in \mathbb{R}_+^N$  such that  $\sum x_i = E$  and for each  $i \in N$ ,

$$x_i = \frac{1}{|N| - 1} \sum_{j \in N \setminus \{i\}} R_i(c_i, c_j, x_i + x_j).$$

Questions are whether such an  $x$  exists, and if it does, whether it is unique. The following theorem states that for most two-claimant rules of interest, both questions have positive answers.

<sup>58</sup>Moreno-Ternero (2002) defines a one-parameter extension of this procedure that produces the awards vectors recommended by the members of the family introduced by Moreno-Ternero and Villar (2001).

<sup>59</sup>This definition is inspired by an idea analyzed in the context of non-transferable utility coalitional games (specifically the class of “hyperplane games”) by Maschler and Owen (1989).

**Theorem 17.** (Dagan and Volij, 1997) *For each resource monotonic two-claimant rule  $R$ , each  $N \in \mathcal{N}$ , and each  $(c, E) \in \mathcal{C}^N$ , there is a unique  $x \in \mathbb{R}_+^N$  such that  $\sum x_i = E$  and for each  $i \in N$ ,*

$$x_i = \frac{1}{|N| - 1} \sum_{j \in N \setminus \{i\}} R_i(c_i, c_j, x_i + x_j).$$

### 6.5. Merging and splitting claims

We consider next a property pertaining to the possibility that a group of agents may consolidate their claims and appear as a single claimant, or conversely, that a given claimant may divide his claim and appear as several claimants. It says that no such consolidation or division should ever be beneficial. It was first studied in the present context by O'Neill (1982).<sup>60</sup>

**No advantageous merging or splitting.** For each pair  $\{N, N'\}$  of elements of  $\mathcal{N}$  such that  $N' \subset N$ , each  $(c, E) \in \mathcal{C}^N$ , and each  $(c', E') \in \mathcal{C}^{N'}$ , if  $E' = E$  and there is  $i \in N'$  such that  $c'_i = c_i + \sum_{N \setminus N'} c_j$ , and for each  $j \in N' \setminus \{i\}$ ,  $c'_j = c_j$ , then  $R_i(c', E') = R_i(c, E) + \sum_{N \setminus N'} R_j(c, E)$ .

This property is satisfied by the proportional rule, but not by any of the other rules that we have seen. In fact we have:

**Theorem 18.** (O'Neill, 1982; Chun, 1988a; de Frutos, 1999; Ju and Miyagawa, 2002<sup>61</sup>)

<sup>60</sup>O'Neill uses the name "strategy-proofness". Banker (1981) considers the stronger requirement that the merging of two agents should not affect the amounts awarded to the others. He studies the property for a wider class of problems in which the sum of the claims is not related to the amount to divide.

<sup>61</sup>This multiple attribution is because these various authors did not all work exactly with the model we considered, and that certain axioms in early characterizations were later shown not to be independent. O'Neill (1982) also imposes the self-explanatory *agent-by-agent claims continuity at at least one point*, as well as *dummy*, *anonymity*, and *limited consistency*. Chun (1988a,b) considers generalized rules and shows that *dummy* (a generalized rule may or may not satisfy this axiom) and *limited consistency* are redundant. He derives a characterization of the proportional rule by exploiting certain logical relations between these axioms (*no advantageous merging or splitting* implies *no advantageous transfer*; *anonymity* and *no advantageous merging or splitting* together imply *dummy* and *limited consistency*), and obtains it as a corollary of a characterization of the proportional rule on the basis of *no advantageous transfer*. That *anonymity* and *continuity* are not needed either is established by de Frutos (1999) for a notion of a rule that does not include *claims boundedness*. Ju and Miyagawa (2002) establish uniqueness of the proportional rule for rules as we defined them on the basis of *no advantageous merging or splitting* alone. Banker (1981) obtains a closely related result based on his strengthening of *no advantageous merging or splitting* mentioned in footnote 60.

*The proportional rule is the only rule satisfying no advantageous merging or splitting.*<sup>62</sup>

In some situations, it may be desirable to forbid the merging of claims but difficult to monitor such operations, and in others the same may be true for the splitting of claims. It is therefore natural to search for rules that satisfy either *no advantageous splitting* (if an agent is replaced by several agents whose claims add up to his, then the sum of what they receive should be at most as large as what he previously received on his own), or *no advantageous merging* (if several agents are replaced by one agent whose claim is equal to the sum of theirs, this agent should receive at most as much as the sum of what they previously received). These properties are studied by de Frutos (1999) and Ju (2001), who search for *consistent* rules satisfying either one of them. Their findings are summarized in the following theorem, which builds on Theorem 14.

**Theorem 19.** (de Frutos, 1999; Ju, 2001) (a) *If a parametric rule satisfies no advantageous merging, then it is a parametric rule relative to a function  $f$  that is sub-additive in its first argument for each value of the parameter  $\lambda$ .*

(b) *If instead, it satisfies no advantageous splitting, then it is a parametric rule relative to a function  $f$  that is super-additive in its first argument for each value of the parameter  $\lambda$ .*

## 6.6. Operators

The claims truncation and attribution of minimal right operators are significantly more disruptive of the properties of rules just formulated for the variable-population version of our model than the duality and convexity operators are. The duality operator preserves *replication invariance*, *consistency* and its *converse*, but not *population monotonicity*. *No advantageous merging* and *no advantageous splitting* are dual properties. The convexity operator preserves *population monotonicity*, *no advantageous merging*, *no advantageous splitting*, and *replication invariance*, but neither *consistency* nor its *converse* (Thomson and Yeh, 2001).

## 6.7. Multiple parameter changes

In the previous sections, we have allowed only one of the parameters entering the description of problems to change, but simultaneous changes in several of these

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<sup>62</sup>Curiel et al. (1987) show that the adjusted proportional rule is the only rule satisfying *invariance under claims truncation*, *minimal rights first*, *equal treatment of equals*, and a weak version of *no advantageous merging or splitting*, obtained by restricting its applications to problems for which all minimal rights are 0 and no claim is greater than the amount to divide.

parameters are just as plausible. Chun (1999) considers the possibility that population and resource change together, or that claims and resource change together, and formulate solidarity requirements appropriate in such circumstances. He also formulates a separability requirement stating that if the claims of some group of agents and the resource change together but the aggregate amount received by the agents with fixed claims is unchanged, then the award to each of these agents should also be unchanged. He describes the logical relations between all of these properties and derives characterizations of the parametric family as corollaries of Theorem 14.

## 7. Strategic models

Here, we present a variety of strategic models superimposed on our basic claims problem. Let  $N$  be a fixed population of claimants and  $(c, E) \in \mathcal{C}^N$ .

In the game formulated by O'Neill (1982), each agent specifies particular parts of the amount to divide, or “units”, as his claim, and any unit that is claimed by several agents is divided equally among them. Therefore, the less overlap there is between what an agent claims and what others claim, the more he receives. The following theorem collects the basic facts about this game,  $\Gamma^O(c, E)$ . Nash equilibria exist and, interestingly, the distribution of claims at equilibrium is a “dual” of O'Neill's extension of Ibn Ezra's method. In the two-claimant case, a unique equilibrium awards vector is obtained, which is that given by concede-and-divide:<sup>63</sup>

**Theorem 20.** (O'Neill, 1982) *For each claims problem  $(c, E) \in \mathcal{C}^N$ , the game  $\Gamma^O(c, E)$  has at least one Nash equilibrium. Any Nash equilibrium is such that there is  $k \in \mathbb{N}$  such that each unit is claimed by exactly  $k$  or  $k + 1$  claimants.*

In the game defined by Chun (1989), agents propose rules instead of awards vectors. Rules are required to satisfy *order preservation*. A sequential revision procedure is defined as follows: the various rules proposed by all the agents are applied to the problem at hand and the claim of each agent is replaced by the maximal amount awarded to him by any one of them. The rules are applied to the problem so revised and a second revision is performed, and so on. The outcome function is defined by taking the limit point of this process, if it exists.<sup>64</sup> Chun (1989) shows that existence is guaranteed, and that in this “game of rules”,  $\Gamma^C(c, E)$ , if the agent with the smallest claim announces the constrained equal awards rule, then for each agent, the sequence of awards calculated by the rule he announced converges to what he would receive under the application of the constrained equal awards rule. A consequence of this result is the following characterization of the unique Nash equilibrium outcome of the game.

**Theorem 21.** (Chun, 1989) *For each claims problem  $(c, E) \in \mathcal{C}^N$ , the game  $\Gamma^C(c, E)$*

<sup>63</sup>To solve the non-uniqueness problem, O'Neill first shows that the set of equilibrium payoffs is a simplex, and he then suggests selecting its center.

<sup>64</sup>This game is inspired by a similar procedure developed by van Damme (1986) for bargaining games.

has a unique Nash equilibrium outcome, which is the awards vector selected by the constrained equal awards rule.

A similar conclusion holds if rules are required to satisfy the first part of *order preservation* and *regressivity*.

A “dual” game can be defined in which, at each stage, each agent is awarded the minimal amount that any of the rules chosen by the various claimants assigns to him. Each claim is then adjusted down by that amount, and the amount to divide is decreased by the sum of the amounts awarded. The process is repeated. This game proposed and is studied by Herrero (2001). A result parallel to Theorem 21 holds, but this time the rule that emerges is the constrained equal losses rule: the game has a unique Nash equilibrium outcome, which is the awards vector selected by this rule for the problem at hand.

Sonn (1992) studies a game of demands similar to the game originally formulated by Chae and Yang (1988) (in their extension of Rubinstein, 1982) for bargaining games and characterizes its subgame perfect equilibria. In this game, player 1 proposes an amount to player 2. If player 2 accepts, he leaves with it and player 1 then proposes an amount to player 3, who again has the choice of leaving with it. If, at some point, a player rejects the offer made to him, the next stage starts with his making an offer to the next player, player 1 being moved to the end of the line. The process continues until only one player is left. The constraint is imposed on offers that no agent should ever be offered an amount greater than his claim or the amount that remains to be distributed. Let  $\Gamma^S(c, E)$  denote the game just defined. In the proof of the following result, consistency and monotonicity properties of certain solutions to bargaining games play an important role.

**Theorem 22.** (Sonn, 1992) *For each claims problem  $(c, E) \in \mathcal{C}^N$ , as the discount factor of future utilities goes to one, the limit of payoff vectors of the game  $\Gamma^S(c, E)$  converges to the awards vector selected by the constrained equal awards rule.*

Serrano (1995) makes use of the *consistency* of the nucleolus as a solution to coalition games to construct for each claims problem a sequential game whose subgame perfect equilibrium outcome is the nucleolus of the associated coalitional game. This result can be extended to the class of *resource monotonic* and *consistent* rules as follows. Assume that a two-claimant rule  $R$  has been selected, and consider the  $n$ -claimant game  $\Gamma^R(c, E)$  in which the agent with the highest claim proposes a division of the amount available, and each of the other agents can (i) either accept his proposed share, in which case he leaves with it, or (ii) rejects it, in which case he leaves with what the two-claimant rule would recommend for him in the problem that the proposer and he would face if they had to divide the sum of the amounts that the proposer proposed for himself and for the agent. The proposer leaves with the difference between the amount available and the sum of the amounts that the accepters accepted and the adjusted amounts rejecters took. Then the game is played again among all the rejecters. For the statement of the next theorem, we need the concept of a *conditionally strictly resource monotonic* rule: it is a rule such that if the amount available increases, then any agent who is not already receiving his claim should receive more.

**Theorem 23.** (Dagan et al., 1997) *Let  $R$  be a resource monotonic and super-modular rule. Then, for each claims problem  $(c,E) \in \mathcal{C}^N$ , the game  $\Gamma^R(c,E)$  has a unique subgame perfect equilibrium outcome, at which each agent receives what the consistent extension of  $R$  recommends. All of the equilibria are coalition-proof<sup>65</sup> if and only if the rule is conditionally strictly resource monotonic.<sup>66</sup>*

The outcome function as specified is not feasible out of equilibrium, but it can be made feasible without the result being affected.

Note that calculating the outcome requires the planner's knowledge of the claims. In a follow-up contribution focusing on taxation problems, Dagan et al. (1999) study the case when incomes are unknown to the planner and can be misrepresented. They impose the natural restriction that only downward misrepresentation is possible, and construct a game form that implements any *consistent* and *strictly claims monotonic* rule in subgame perfect equilibrium.

In the game defined by Corchón and Herrero (1995), agents propose awards vectors that are bounded by claims. The proposals are combined, by means of a “compromise function”, so as to produce a final outcome. The authors establish necessary and sufficient conditions on a two-claimant rule for it to be implementable in dominant strategies: the rule should be strictly increasing in each claim, and the amount received by each agent should be expressible as a function of his claim and the difference between the amount available and the claim of the other agent. Implementation can be achieved by a simple averaging of proposals. For the  $n$ -person case, the results are largely negative, however, at least when the averaging method is used.<sup>67</sup>

In summary, we see that a number of the rules that we had arrived at on the basis of axiomatic considerations have been provided additional support by taking the strategic route.

## 8. Extensions of the basic model

In this section we discuss extensions of the model to surplus-sharing and to situations where utility is non-transferable.

Estate division problems can be generalized in different ways. First, as O'Neill (1982) notes, the number of documents in which amounts are bequeathed need not be equal to the number of heirs. Also, in each document, more than one heir may be named.

<sup>65</sup>Bernheim et al. (1985).

<sup>66</sup>For a *strictly resource monotonic* rule, the identity of the proposer is immaterial.

<sup>67</sup>Landsburg (1993) studies a problem of manipulation in which manipulation is costly. The cost of misrepresenting one's claim is given by a function having the property that the greater the extent of the manipulation, the greater is the cost incurred. In the special case in which the claims add up to the estate, he finds that there is a single generalized rule giving agents the incentive to report truthfully. It is the generalized rule that equates losses. In Sertel's (1992) game, the strategic opportunity of an agent is to transfer a fraction of his claim to the other player (there are two players), payoffs being calculated by applying the Nash bargaining solution to a certain bargaining game associated with the claims problem. He shows that, at equilibrium, the two players receive the awards concede-and-divide would select.

Alternatively, each document may specify a complete division of the estate among all the heirs.

A study of the case when the data of the problem are integers and awards vectors are required to have integer coordinates is due to Moulin (2000).

We have already introduced the notion of a generalized rule: such a rule may violate *non-negativity* and *claims boundedness*. All the remaining theorems in this section concern generalized rules. To simplify notation we write  $n$  instead of  $|N|$ .

**Theorem 24.** (Chun, 1988a<sup>68</sup>) *For  $n \geq 3$ . A generalized rule  $R$  satisfies continuity, anonymity, and no advantageous transfer if and only if there exists a continuous function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that for each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,*

$$R_i(c, E) = \frac{c_i}{\sum c_i} E - \frac{1}{\sum c_i} \left\{ (n-1)c_i - \sum_{N \setminus \{i\}} c_j \right\} g(\sum c_i, E).$$

The family described in Theorem 24 includes the proportional rule and the equal awards generalized rule (for  $g(c, E) = 0$  and  $g(c, E) = E/n$ , respectively).

**Theorem 25.** (Chun, 1988a) *A generalized rule  $R$  satisfies continuity, anonymity, and resource linearity if and only if there exist continuous functions  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  that are invariant with respect to permutations of their last  $n - 1$  arguments, and such that for each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ ,*

$$R_i(c, E) = \frac{E}{n} + \frac{E}{n} \left[ (n-1)h(c_i, c_{-i}) - \sum_{j \in N \setminus \{i\}} h(c_j, c_{-j}) \right] + \frac{1}{n} \left[ (n-1)g(c_i, c_{-i}) - \sum_{j \in N \setminus \{i\}} g(c_j, c_{-j}) \right].$$

**Theorem 26.** (Chun, 1988a) *A generalized rule  $R$  satisfies continuity, anonymity, and resource additivity if and only if there exists a continuous function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  that is invariant with respect to permutations of its last  $n - 1$  arguments, and such that for each  $(c, E) \in \mathcal{C}^N$  and each  $i \in N$ :*

$$R_i(c, E) = \frac{E}{n} + \frac{E}{n} \left\{ (n-1)h(c_i, c_{-i}) - \sum_{N \setminus \{i\}} h(c_j, c_{-j}) \right\}.$$

A corollary of this theorem is another characterization of the proportional rule. Here too, it is obtained by requiring that the generalized rule actually be a rule. In fact, it suffices to require that  $R(c, E) = c$  if  $\sum c_i = E$ , or that the generalized rule be *self-dual*. Alternatively, *continuity* can be replaced by *resource monotonicity*.

Recent studies of on the subject are by Bergantiños and Méndez-Naya (1977, 2001); Ching and Kakkar (2001) and Naumova (2002).

<sup>68</sup>Chun (1988a) notes that a similar result would obtain if *no-transfer paradox* were imposed instead of *continuity*, but then  $g$  would not have to be continuous.

### 8.1. Surplus-sharing

Closely related to claims problems are surplus-sharing problems (Moulin, 1985a, 1987). Such a problem is a pair  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$  such that  $E \geq \sum c_i$ , where  $c_i$  is interpreted as the investment in a joint venture made by agent  $i \in N$  and the difference  $E - \sum c_i$  is the *surplus* generated by this venture. How should it be divided among the investors? Moulin characterizes one-parameter families of surplus-sharing rules that contain as particular cases equal sharing and proportional sharing. One of the auxiliary axioms he imposes is *homogeneity* (see above). Pfingsten (1991) describes how the class of admissible rules enlarges when *homogeneity* is not required. Monotonicity and strategic issues for this model are studied by Chun (1988a, 1989).

An even more general class of problems consists of pairs  $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ , in which no restriction is imposed on the value of  $E$  as compared to  $\sum c_i$ . This class includes both claims problems and surplus sharing problems as particular cases. Rules defined for it can be easily obtained by piecing together rules to adjudicate conflicting claims and rules to divide surpluses. An analysis of this class is carried out by Herrero et al. (1999).

Finally, we could consider the class just defined but without claims being commensurable with the amount to divide. For instance,  $c_i$  could be interpreted as the “contribution”, the “need”, or the “merit”, of agent  $i$ . In such context, comparing  $\sum c_i$  to  $E$  may not be meaningful. Conditions such as *claims boundedness* may not be meaningful either (see our earlier discussion of *resource additivity*).

Dagan (1994) considers the taxation problem interpretation of the model and proposes a richer formulation that includes constraints on transfers across agents. He characterizes the equal-sacrifice rule, mainly on the basis of *consistency* considerations.

### 8.2. Non-transferable utility problems

In many applications, it is not legitimate to assume that transfers of money among agents is accompanied by one-to-one transfers of utility. When agents are equipped with utility functions over money that are not linear, our previous analysis does not apply. Chun and Thomson (1992) formulate and analyze a class of claims problems in which utility functions are not restricted to be linear. The image in utility space of such a “non-transferable utility claims problem” can also be seen as a bargaining game enriched by the addition of a claims vector outside of the feasible set. Such problems are “bargaining problems with claims”. Chun and Thomson offer several characterizations of the proportional rule: in this setting, this is the solution that selects the maximal feasible point on the line connecting the origin to the claims vector (Fig. 5).

Other solutions have been defined for this model (Bossert, 1992; Herrero, 1993; Marco, 1994, 1995a,b; Herrero and Marco, 1992, 1993). The main alternative to the proportional rule is the “extended claim egalitarian solution” which selects for each problem the payoff vector at which the utility losses from the claims point are equal across agents, subject to the requirement that no agent receives less than 0. This point can be obtained in either one of the following alternative ways. (i) First, select the maximal feasible point on the path defined by moving down from the claims point in such a way that all agents whose utilities are still positive experience equal losses, and

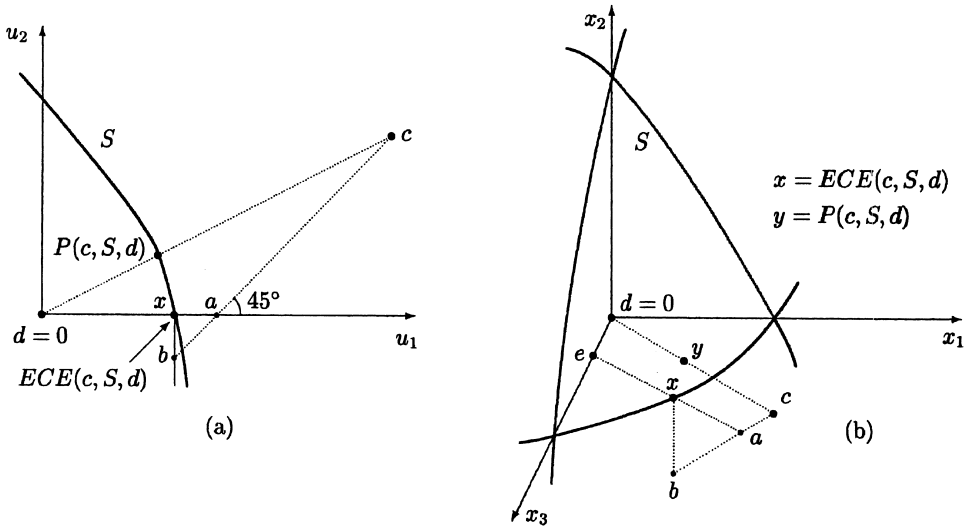


Fig. 5. Non-transferable utility claims problems. In a non-transferable utility claims problem the boundary of the feasible set is not a straight line of slope  $-1$ . Two solutions for such problems are illustrated for  $n = 2$  in panel (a) and  $n = 3$  in panel (b). For the proportional solution,  $P$ , payoffs are chosen proportional to claims. For the extended claim-egalitarian solution,  $ECE$ , payoffs are obtained by imposing equal losses from the claims point, subject to the constraint that no agent should end up with a negative utility.

all other agents receive 0 (in Fig. 5a this is the path  $c,a,0$ , and in Fig. 5b it is the path  $c,a,e,0$ ; in either case, the resulting recommendation is  $x$ ). (ii) The other definition is in two steps. Find the maximal feasible point of equal losses in the “comprehensive hull” of the individually rational part of the problem (this is the set of points in  $\mathbb{R}^N$  that are dominated by some feasible point that dominates the origin; in Fig. 5a and b, this maximal point is  $b$ ). Then, set equal to 0 the utility of each agent whose individuality rationality constraint is violated (this leads us back to  $x$ ). Further adjustments in both the proportional and the extended claim-egalitarian solutions are needed to obtain Pareto-optimal outcomes. For that purpose, a lexicographic operation can be performed (Marco, 1995a), parallel to operations that have been found useful in the context of bargaining to handle a similar difficulty (Chun and Peters, 1991).

### 8.3. Model with group constraints

A class of problems incorporating constraints on what groups can achieve is formulated and studied by Bergantiños and Sanchez (2002). They imagine situations in which an upper bound is given not only on what each individual claimant can receive but also on what each group of claimants can receive (Bergantiños and Sanchez (1999) describes an application of the model). They define an extension of the constrained equal awards rule and offer an axiomatization of it. A related model is examined by Carpenté et al. (2002), who add to the specification of a claims problem a partition of the set of claimants, the members of each group in the partition being related by their claims. They

too propose an extension of the constrained equal awards rule, and they provide several axiomatizations of it.

#### 8.4. *Experimental testing*

In the last 20 years, a considerable literature has emerged concerned with the experimental testing of economic theories. Is the intuition we have about rules and axioms also that of experimental subjects? Do they resolve cooperative games and play strategic games according to the behavior that we postulate in our formal models? And how do the answers depend on the context? A first study of claims problems addressing these issues is due to Ponti et al. (2002).

### 9. Conclusion

Although claims problems are among the simplest that one may encounter, we have discovered that the model is surprisingly rich. Axiomatic analysis has been of great help in providing support for the rule that is the most commonly used in practice, namely the proportional rule, but it has provided support for several other rules that have played a role in practice and theory, as well as uncovered new rules. Together with the recent studies of claims problems as strategic games, we now have an incomparably better understanding of the problem than just a few years ago.

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