A monotonic and merge-proof rule in minimum cost spanning tree situations*

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March 6, 2015

Abstract

We present a new model for cost sharing in minimum cost spanning tree problems, so that the planner can identify the agents that merge. Under this new framework, and as opposed to the traditional model, there exist rules that satisfy *merge-proofness*. Besides, by strengthening this property and adding some other properties, such as *population-monotonicity* and *solidarity*, we characterize a unique rule that coincides with the weighted Shapley value of an associated cost game.

Keywords: Minimum cost spanning tree problems, cost sharing, core selection, cost-monotonicity, merge-proofness, weighted Shapley value.

JEL Classification Codes: C71, D61, D63, D7.

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*Latest version at: `http://webs.uvigo.es/vidalpuga/`. Financial support by the Spanish Ministerio de Ciencia e Innovación through grant ECO2011-23460, and Xunta de Galicia through grants 10PXIB362299PR and EM2013/013, is gratefully acknowledged.

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1 Introduction

Problems in which supply network formation is involved have been widely studied in the literature from two different points of view. On one hand, the design of efficient algorithms and the computational complexity. On the other hand, the cost sharing of the network.

In situations where the supply can be done either directly or though other nodes, the optimal network is a tree. Examples include water supply, electricity, cable television, Internet, and so on.

In order to study these situations, two steps should be considered: in the first one, agents will construct a minimal cost tree, *mt* for short. There exist several algorithms for building a *mt*. Two of them are provided by Kruskal (1956) and Prim (1957), respectively. Once we have constructed a *mt*, the second step consists of dividing its cost between the nodes that benefit from it. In order to do that, a rule will be used.

In this paper, we focus on the cost sharing aspect. In particular, we study minimum cost spanning tree (*mcst*) situations. A group of agents, located at different geographical points, want some particular service or good which can only be provided by a common supplier, called the *source*. Agents can be served through connections which entail some cost and they do not care about whether they are connected directly or indirectly to the source.

There is a large literature on the cost-sharing related problem. Bird (1976) uses Prim's algorithm and proposes a cost allocation rule, the so-called Bird rule. He also associates a cooperative game with each *mcst* problem. Granot and Huberman (1981, 1984) study the core and the nucleolus of this cooperative game. Feltkamp et al. (1994) use Kruskal's algorithm in order to define a rule, the folk rule, which has been redefined and characterized in Bergantiños and Vidal-Puga (2007a,b). The definition of the folk rule in Bergantiños and Vidal-Puga (2007b) is as the Shapley value of a particular game associated to the problem. Other authors also use the Shapley value in order to define rules. Kar (2002) studies the Shapley value of the cooperative game defined by Bird (1976). Trudeau (2012) defines the cycle-complete rule as the Shapley value of a another particular game. Dutta and Kar (2004) propose another different rule.

A relevant approach is to study which desirable properties are satisfied by the different rules. Properties should help a planner to compare different rules and to decide which rule is preferred in a particular case.

In this paper, we focus on three important classes of properties. The first class is based on the property of *core selection*. This property states that no group of agents should subsidize the other agents, paying more than the cost of connecting themselves to the source. This is a very relevant property in the economic literature. A typical drawback is that, in general, the core may be empty. However, it happens to be always non-empty in the game associated to a *mcst* problem. In fact, most of the rules proposed in the literature satisfy core-selection, with the remarkable exception of the one derived from the Shapley value of the associated game (see Kar (2002)). A stronger version of core selection is *population monotonicity*, which requires that the cost allocated to any agent will not decrease if new agents join the society. Population monotonicity in *mcst* problems has been studied by Bergantiños and Gómez-Rúa (2010), Bergantiños and Vidal-Puga (2007a, 2009), Bogomolnaia and Moulin (2010), Lorenzo and Lorenzo-Freire (2009), and Norde et al. (2004).

The second class is based on the property of *cost monotonicity*. This property implies that the cost allocated to some node will not increase if the cost of a link involving this node goes down, nothing else changing. Hence, a violation of this appealing property could disincentive the agents to reduce the costs of constructing links (see Dutta and Kar (2004)). A stronger version of cost monotonicity requires that the cost allocated to any agent will not increase if the cost of any link (involving this player or not) goes down, nothing else changing. Hence, this strong version would also prelude the agents to sabotage the construction of any link. This stronger version of cost monotonicity in *mcst* problems has been studied, among others, in Bergantiños and Gómez-Rúa (2010), Bergantiños and Vidal-Puga (2007a, 2009), Bogomolnaia and Moulin (2010), and Trudeau (2012).

There have been studied several rules that satisfy both population monotonicity and cost monotonicity. In particular, the folk rule is the only of these rules that satisfies both properties (see Bergantiños and Vidal-Puga (2008)). For a characterization of rules satisfying both properties, see Bergantiños and Vidal-Puga (2015).

The third class is based on the properties of *split* and *merge-proofness.* The split-proofness property implies that one node should not have incentives to split into two or more different nodes. The merge-proofness property implies that two or more nodes should not have incentives to merge into a single node. These properties are relevant in situations where the identity of the nodes is ambiguous. For example, different departments on a University campus can be already connected by an internal network. In case they want to be connected to a wider supply network, should they be considered as a single node (the campus) or as several different nodes (the departments) connected at zero cost? Other examples are the different shops at the mall, apartments on a building, or houses in a residential area.

Merge-proofness preludes the agents to build an inefficient network, as next example shows:

Example 1.1 *Consider three agents located respectively at nodes* 1*,* 2 *and* 3*. The connection costs are depicted in Figure 1 (left). Figure 1 (right) represents the same situation after agents* 2 *and* 3 *merge, paying cost* 8*. This merging is inefficient, because the three agents end up paying no less than* $8 + 28 = 36$*, whereas the initial problem has an optimal tree with cost* 34*. Assume a rule assigns* $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ *in the initial problem, and* $\Phi' = (\Phi'_1, \Phi'_2)$ *in the second one. Then, a merge-proof rule should satisfy* $\Phi_2 + \Phi_3 \leq 8 + \Phi'_2$*. Inefficiency due to a previous merge of nodes* 2 *and* 3 *is then avoided.*

The properties of split and merge-proofness have been studied in many contexts related to cost sharing problems. In a context where the agents

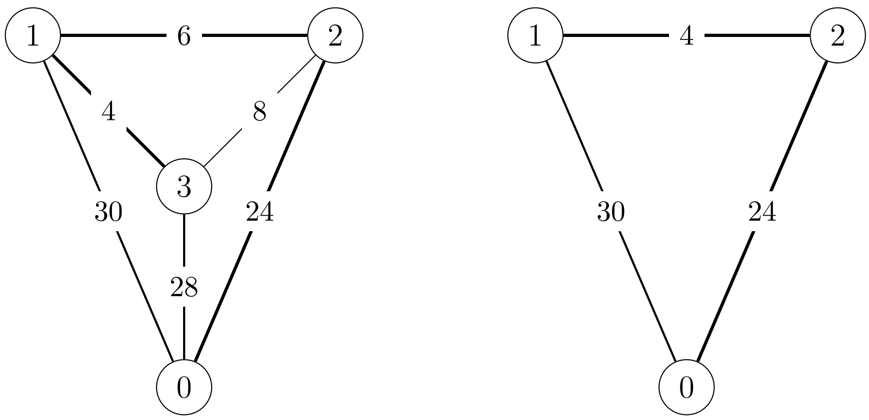

Figure 1: Merging of nodes 2 and 3.

consume arbitrary quantities of possibly different goods, Sprumont (2005) characterizes the Aumann-Shapley cost sharing method, used to distribute the total generated cost that ensures that agents never find it profitable to split or to merge their consumptions. O'Neill (1982), Chun (1988), De Frutos (1999), Moulin (2002), Ju (2003), and Ju et al. (2007) study split and merge-proofness in claim and bankruptcy problems. Moulin (2007, 2008) also studies split and merge-proofness in the context of job scheduling. Merging in exchange economies has been studied for several solution concepts by Hart (1974), Postlewaite and Rosenthal (1974), Maschler (1976), Legros (1987) and Rosenmüller and Sudhölter (2004). Merging and splitting in cooperative games have also been studied by Knudsen and Østerdal (2012) and references herein.

In the context of *mcst* problems, the folk rule satisfies split-proofness and, moreover, it is not difficult to derive a split-proof rule from a cost-monotonic one. However, this is not the case with merge-proofness. Under domain restrictions[1], the Bird rule satisfies merge-proofness (see Özsoy (2006), Athanassoglou and Sethuraman (2008), and Gómez-Rúa and Vidal-Puga (2011)). The Bird rule is, moreover, the only relevant rule defined in the literature that satisfies it (see Gómez-Rúa and Vidal-Puga (2011)).

[1]For example, assuming all the costs are different.

However, in the most general setting, no rule satisfies merge-proofness, as the next example shows:

Example 1.2 (Özsoy (2006)) *Assume we have three agents located, respectively, at nodes 1, 2 and 3. The connection cost between each agent and the source is* 1*, and the connection cost between any pair of agents is* 0*. The minimal cost is* 1 *and hence any rule* Φ *should satisfy* $\Phi_1 + \Phi_2 + \Phi_3 = 1$*. Assume w.l.o.g.* $\Phi_3 \geq \max\{\Phi_1, \Phi_2\}$*. Now, if players 1 and 3 join and appear as agent 1 alone, the planner would face the same problem as if players 2 and 3 join and appear as agent 2 alone. Then, a merge-proof rule should assign to player 1 (say* Φ_1'*) at least as much as to players 1 and 3 in the original problem, whereas it should assign to player 2 (say* Φ_2'*) at least as much as to players 2 and 3 in the original problem. This means* $\Phi_1 + \Phi_3 \leq \Phi_1'$ *and* $\Phi_2 + \Phi_3 \leq \Phi_2'$*, which implies* $\Phi_1 + \Phi_2 + 2\Phi_3 \leq \Phi_1' + \Phi_2'$*. Since* $\Phi_1 + \Phi_2 + \Phi_3 = 1$ *and* $\Phi_1' + \Phi_2' = 1$*, we deduce* $\Phi_3 \leq 0$*, which is impossible because* $\Phi_3 \geq \max\{\Phi_1, \Phi_2\}$ *and* $\Phi_1 + \Phi_2 + \Phi_3 = 1$*.*

The key issue in the previous example is that the planner has no way to know whether agent 3 has merged with agent 1 or with agent 2. This assumption is necessary in situations where the agents may use multiple replicas without being detected, as for example the case of users of a web page. However, this may not be a reasonable assumption in many other situations. In the *mcst* model, one may think in the case of departments in a campus or apartments in a building. In case the planner knows which mergers may have taken place, it is not difficult to derive a merge-proof rule[2]. On the other hand, it is not clear whether a merge-proof rule could also satisfy core selection and cost monotonicity[3].

In this paper, we model the *mcst* situation in such a way that the planner knows which mergers take place. If some agents merge and present themselves to the planner in this way, she should solve a situation where the nodes

[2]For example, charging all the cost to the merging agents.

[3]Charging all the cost to the merging agents will clearly not satisfy core selection.

are formed by the union of several agents. We refer to this new problem as a (*mcst*) *situation*. A *mcst* situation generalizes the classical *mcst* problem.

Under this model, all the rules presented in the literature fail at least one of the properties. However, we propose a new rule that satisfies all of them, even the stronger versions. This rule is the weighted Shapley value of a particular cooperative game[4]. We also propose a characterization result of this rule with these and other properties. The other properties used in the characterization are *efficient merging*, *piece-wise additivity*, *symmetry* and *positivity*.

Positivity says that each node should pay at least zero. It has been used, in the context of minimum cost spanning tree problems, by Bergantiños and Vidal-Puga (2009). Piece-wise additivity is a weaker version of additivity. Additivity and symmetry are standard properties. They are used in the classical characterization of the Shapley value (Shapley (1953)). Different versions of piece-wise additivity property have also been used in the context of minimum cost spanning tree problems by Branzei et al. (2004), Tijs et al. (2006), Bergantiños and Vidal-Puga (2009), Bergantiños et al. (2010), and Hougaard et al. (2010). Piece-wise additivity says that when the same network is optimal for two different cost matrices, then the cost-sharing is additive in the cost function.

Efficient merging, on the other hand, is related to the properties of split and merge-proofness. Even though merge-proofness precludes agents to build an inefficient network (as in Example 1.1), this concern is no longer relevant when the nodes that merge are already the closest ones (as in Example 1.2). Efficient Merging says that these nodes should not find harmful to join in advance in these cases.

The paper is organized as follows: In Section 2 we present the model. In Section 3 we describe some desirable properties of the rules. In Section

[4]In the context of pricing traffic demand in a spanning network, Moulin (2014) also finds the weighted Shapley value of a cooperative game to satisfy the so-called routing-proofness. This property is related to split-proofness, since it precludes the agents to get advantage by reporting to be several different users along a path.

4 we define a new rule, prove that it satisfies all these properties and characterize it with some of them. In Section 5 we prove that this rule is the weighted Shapley value of a particular transferable-utility game. In Section 6 we present some conclusions.

2 The model

Let $N = \{1, \ldots, n\}$ be a finite set of potential agents, $n > 2$, and let 0 be a special point called the *source*. Let $N_0 = N \cup \{0\}$. A *minimum cost spanning tree problem* is determined by a cost function $c : N_0 \times N_0 \to \mathbb{R}_+$ that assigns a non-negative cost to each pair $(i, j) \in N_0 \times N_0$. We assume $c(i, i) = 0$ for all $i \in N_0$ and $c(i, j) = c(j, i)$ for all $i, j \in N_0$.

We assume that some agents in N want to be connected to the source, and they are indifferent between connecting directly or through other agents. The cost of direct connection between any pair of agents, or between any agent and the source, is given by c. Moreover, some groups of agents can be connected in advance, so that they behave as a single nodes. Hence, these nodes are groups of agents.

A *minimum cost spanning tree situation*, or simply a *situation*, of a cost function c, is a pair $\left(\mathcal{P}, c^{\mathcal{P}}\right)$ where $\mathcal{P} = \{P_0, P_1, ..., P_m\}$ is a subset of mutually exclusive subsets of N_0 (i.e. $P_r \cap P_s = \emptyset$ when $r \neq s$) with $0 \in P_0$, and $c^{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \to \mathbb{R}_+$ is a function that assigns to each pair of (unsorted) subsets in $\mathcal{P}$ the minimum cost of connecting any pair of agents of their respective group, that is, $c^{\mathcal{P}}(P_r, P_s) = \min_{i \in P_r, j \in P_s} c(i, j)$ for all $P_r, P_s \in \mathcal{P}$. Each $P_r \in \mathcal{P}$ is a *node*.

For simplicity, we write $(\mathcal{P}, c)$ instead of $\left(\mathcal{P}, c^{\mathcal{P}}\right)$. Moreover, and when there is no possible confusion, we write r, s instead of P_r, P_s, and so on. Notice that we write 0 instead of P_0, since it plays the role of the source in a given situation.

We denote as $E^{\mathcal{P}} = \{\{r, s\} : r, s \in \mathcal{P}, r \neq s\}$ the set of *edges* in $\mathcal{P}$. A *graph* g in $\mathcal{P}$ is a subset of $E^{\mathcal{P}}$. The *cost* of a graph g in $(\mathcal{P}, c)$ is defined as

$$c(g) = \sum_{\{r,s\} \in g} c^{\mathcal{P}}(r, s).$$

We denote as $G^{\mathcal{P}}$ the set of graphs in $(\mathcal{P}, c)$.

A *path* in $(\mathcal{P}, c)$ is a sequence $(r_0, ..., r_k)$ of different nodes in $\mathcal{P}$. In particular, we say that $(r_0, ..., r_k)$ is a *path between* r_0 and r_k. We say that a path $(r_0, \ldots, r_k)$ is *in* a graph g in $\mathcal{P}$ if $\{r_{l-1}, r_l\} \in g$ for all $l = 1, ..., k$.

A *spanning graph* in $\mathcal{P}$ is a graph g in $\mathcal{P}$ such that for all $r, s \in \mathcal{P}$, there exists a path in $\left(\mathcal{P}, c^{\mathcal{P}}\right)$ between r and s. We denote as $SG^{\mathcal{P}}$ the set of spanning graphs in $\mathcal{P}$.

A *rule* Φ is a function that assigns to each $(\mathcal{P}, c)$ a vector $\Phi(\mathcal{P}, c) \in \mathbb{R}^{\mathcal{P} \setminus \{0\}}$ satisfying

$$\sum_{r \in \mathcal{P} \setminus \{0\}} \Phi_r(\mathcal{P}, c) = \min_{g \in SG^{\mathcal{P}}} c(g).$$

A *spanning tree* in $\mathcal{P}$ is a graph t in $\mathcal{P}$ such that for all $r, s \in \mathcal{P}$, there exists a unique path in $\mathcal{P}$ between r and s. If t is a spanning tree, we usually write $t = \{(r, r^0)\}_{r \in \mathcal{P} \setminus \{0\}}$, where r^0 represents the first node in the unique path in t from r to the source. We denote as $ST^{\mathcal{P}}$ the set of spanning trees in $(\mathcal{P}, c)$.

Since $c(r, s) \geq 0$ for all $(r, s) \in E^{\mathcal{P}}$, it is clear that we can replace $SG^{\mathcal{P}}$ with $ST^{\mathcal{P}}$ in the definition of rule. Such a *minimal cost spanning tree* is called a *minimal tree*, *mt* for short. A minimal tree always exists, even though it is not necessarily unique. We denote as $MT(\mathcal{P}, c)$, $MT^{\mathcal{P}}$ for short, the set of minimal trees in $(\mathcal{P}, c)$. We denote the cost associated with any *mt* on $(\mathcal{P}, c)$ as $m(\mathcal{P}, c)$.

For any $(\mathcal{P}, c)$, a *connected component* is a maximal subset of $\mathcal{P}$ where all the nodes can be connected at zero cost, that is, for any two nodes r, s in the same connected component, there exists a path $(r_0, \ldots, r_k)$ between r and s such that $c(r_{l-1}, r_l) = 0$ for all $l = 1, \ldots, k$. Clearly, the connected components determine a partition $\mathbb{P}$ of $\mathcal{P}$ which includes exactly one set S_0 of nodes that are connected to the source at zero cost. We assume $0 \in S_0$ so that $S_0 \neq \emptyset$.

3 Properties of the rules

In this section we describe the properties that we consider a cost sharing rule should satisfy. Most of them are well-known in the classical model of *mcst* problems and we adapt them to the new context. We also propose a new one. Now we present them in a formal way. Let Φ be a generic rule.

Core Selection: For each $\mathcal{Q} \subset \mathcal{P} \setminus \{0\}$, we have

$$\sum_{r \in \mathcal{Q}} \Phi_r(\mathcal{P}, c) \leq \min_{t \in ST^{\mathcal{Q} \cup \{0\}}} c(t).$$

This property says that no subset of coalitions can find it cheaper to create their own network without the others.

Population Monotonicity: For each $r, s \in \mathcal{P} \setminus \{0\}$, we have

$$\Phi_r(\mathcal{P}, c) \leq \Phi_r(\mathcal{P} \setminus \{s\}, c).$$

This property says that if the population of agents decreases, nobody is better off. Equivalently, if the population of agents increases, nobody is worse off.

It is straightforward to check that Population Monotonicity implies Core Selection.

Cost Monotonicity: For each $i \in P_r \in \mathcal{P}$, $\Phi_r(\mathcal{P}, c)$ is non-decreasing on $c(i, j)$ for all $j \in N_0 \setminus \{i\}$.

This property says that if a connection cost increases for coalition r and the rest of the connection costs remain the same, then coalition r is not better off.

The following property is a stronger version of Cost Monotonicity.

Solidarity: $\Phi(\mathcal{P}, c)$ is non-decreasing on $c(i, j)$ for all $i, j \in N_0$.

This property says that if a connection cost increases and the rest of connection costs remain the same, then no coalition is better off.

It is clear that Solidarity implies Cost Monotonicity.

Positivity: $\Phi(\mathcal{P}, c)$ only takes non-negative values.

This property says that no agent can be compensated by connecting to the source.

Merge-Proofness: For each $\mathcal{Q} \subset \mathcal{P} \setminus \{0\}$ and $g \in SG^{\mathcal{Q}}$, we have

$$\sum_{r \in \mathcal{Q}} \Phi_r(\mathcal{P}, c) \leq \Phi_q((\mathcal{P} \setminus \mathcal{Q}) \cup \{q\}, c) + c(g)$$

where $q = \bigcup_{r \in \mathcal{Q}} r$ (or $P_q = \bigcup_{P_r \in \mathcal{Q}} P_r$).

This property says that no group of coalitions have incentives to join a priori, assuming the cost (given by $c(g)$), to be treated as a single node.

Strong Merge-Proofness: For each $\mathcal{Q} \subset \mathcal{P} \setminus \{0\}$, and $s \in (\mathcal{P} \setminus \mathcal{Q}) \setminus \{0\}$, we have

$$\Phi_s((\mathcal{P} \setminus \mathcal{Q}) \cup \{q\}, c) \leq \Phi_s(\mathcal{P}, c)$$

where $q = \bigcup_{r \in \mathcal{Q}} r$ (or $P_q = \bigcup_{P_r \in \mathcal{Q}} P_r$).

This property says that if a group of coalitions ($\mathcal{Q}$) join in advance in order to be treated as a single node (q), no other coalition (s) will be worse off in the reduced problem.

Strong Merge-Proofness implies Merge-Proofness (see Gómez-Rúa and Vidal-Puga (2011)).

The following property considers the case in which one particular node splits into several nodes, producing a new situation with additional nodes.

Split-Proofness: For each $\mathcal{Q} \subset \mathcal{P} \setminus \{0\}$, we have

$$\Phi_q((\mathcal{P} \setminus \mathcal{Q}) \cup \{q\}, c) \leq \sum_{r \in \mathcal{Q}} \Phi_r\left(\mathcal{P}, c^{0q}\right)$$

where $q = \bigcup_{r \in \mathcal{Q}} r$ (or $P_q = \bigcup_{P_r \in \mathcal{Q}} P_r$) and c^{0q} is the cost function defined by $c^{0q}(i, j) = 0$ for all $i, j \in P_q$, and $c^{0q}(i, j) = c(i, j)$ otherwise.

This property says that no node (q) has incentives to split into several nodes ($\mathcal{Q}$).

Efficient Merging: If there exist two nodes $r, s \in \mathcal{P}\backslash\{0\}$ such that $c\left(r, s\right) = \min_{r',s' \in \mathcal{P}} c\left(r', s'\right)$, then

$$\Phi_{r\cup s}\left(\mathcal{P}^{rs}, c\right) + c\left(r, s\right) \leq \Phi_r\left(\mathcal{P}, c\right) + \Phi_s\left(\mathcal{P}, c\right)$$

where $\mathcal{P}^{rs} = \left(\mathcal{P} \setminus \{r, s\}\right) \cup \{r \cup s\}$.

This property says that if the closest nodes (r and s) are formed by agents, then they should find it optimal to merge. Hence, it avoids disincentives to construct an optimal network.

There exist some relations among these four last properties. Strong Merge-Proofness implies Merge-Proofness (see Gómez-Rúa and Vidal-Puga (2011)). Next Proposition shows that Strong Merge-Proofness and Efficient Merging imply Split-Proofness.

Proposition 3.1 *If a rule satisfies Strong Merge-Proofness and Efficient Merging, then it also satisfies Split-Proofness.*

Proof. Let $\mathcal{Q} \subset \mathcal{P} \setminus \{0\}$. We need to prove, under Strong Merge-Proofness and Efficient Merging, that $\Phi_q((\mathcal{P} \setminus \mathcal{Q}) \cup \{q\}, c) \leq \sum_{r \in Q} \Phi_r(\mathcal{P}, c^{0q})$. We proceed by induction on $|Q|$, the cardinality of Q. For $Q = \{q\}$, it is clear that $(\mathcal{P} \setminus \mathcal{Q}) \cup \{q\} = \mathcal{P}$ and, moreover, $c = c^{0q}$. Hence the result. Suppose now the result holds for $|Q| < \alpha$ with $\alpha > 1$, and assume $|Q| = \alpha$. Fix $r \in Q$ and let $\mathcal{Q}' = \mathcal{Q} \setminus \{r\}$. Let $\mathcal{P}' = (\mathcal{P} \setminus \mathcal{Q}') \cup \{q'\}$, where $q' = q \setminus r$ (or $P_{q'} = P_q \setminus P_r$). It is clear that $(\mathcal{P} \setminus \mathcal{Q}) \cup \{q\} = (\mathcal{P}')^{q'r}$. Hence, $\Phi_q\left((\mathcal{P} \setminus \mathcal{Q}) \cup \{q\}, c\right) = \Phi_q\left((\mathcal{P}')^{q'r}, c\right) = \Phi_q\left((\mathcal{P}')^{q'r}, c^{0q}\right)$. Under Efficient Merging, this is less of equal than $\Phi_{q'}\left(\mathcal{P}', c^{0q}\right) + \Phi_r\left(\mathcal{P}', c^{0q}\right)$. Under the induction hypothesis, $\Phi_{q'}\left(\mathcal{P}', c^{0q}\right) \leq \sum_{s \in \mathcal{Q}'} \Phi_s\left(\mathcal{P}, c^{0q}\right)$. Under Strong Merge-Proofness, $\Phi_r\left(\mathcal{P}', c^{0q}\right) \leq \Phi_r\left(\mathcal{P}, c^{0q}\right)$, and hence the result. ■

For the next property, given an order $\sigma : \{1, \ldots, |E^{\mathcal{P}}|\} \longrightarrow E^{\mathcal{P}}$, we define $\mathsf{C}_\sigma = \left\{x \in \mathbb{R}_+^{E^{\mathcal{P}}} : 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(|E^{\mathcal{P}}|)}\right\}$ as the cone in $\mathbb{R}^{E^{\mathcal{P}}}$ such that the ordering of the coordinates is given by σ. A real valued function F with domain $\mathbb{R}_+^{E^{\mathcal{P}}}$ is *piece-wise additive* if for any σ its restriction to C_σ is additive, i.e. $F(x+y) = F(x) + F(y)$ for all $x, y \in \mathsf{C}_\sigma$.

Piece-wise Additivity: Φ is piece-wise additive as a function with domain $\mathbb{R}_+^{E^{\mathcal{P}}}$. Namely,

$$\Phi(\mathcal{P}, c + c') = \Phi(\mathcal{P}, c) + \Phi(\mathcal{P}, c')$$

for all $c, c' \in \mathsf{C}_\sigma$.

This property provides a vector structure to $\Phi(\mathcal{P}, c)$. The main advantage of a piece-wise additive cost sharing rule is that it is entirely determined by its value over the $|E^{\mathcal{P}}|$-coordinate vectors whose coordinates take exactly two values, one of them positive and the other zero (compare page 302 in Hougaard et al. (2010)).

For the last property, we define symmetric coalitions. Two coalitions $r, s \in \mathcal{P} \setminus \{0\}$ are *symmetric* in $(\mathcal{P}, c)$ if they have the same number of agents and, moreover, $c(r, u) = c(s, u)$ for all $u \in \mathcal{P} \setminus \{r, s\}$.

Symmetry: Symmetric coalitions pay the same.

4 A monotonic and merge-proof rule

Definition of the rule

In order to define our rule, we need some additional notation. Given any $t \in MT^{\mathcal{P}}$ and $r, s \in \mathcal{P}$, let $\overline{c}(r, s)$ denote the maximum cost in the (unique) path between r and s in t. Formally, given $\{r_0, \ldots, r_k\}$ be the (unique) path between $r = r_0$ and $s = r_k$ in t, we define

$$\overline{c}(r, s) = \max_{l=1,\ldots,k} \{c(r_{l-1}, r_l)\}.$$

This cost function $\overline{c}$ determines the *irreducible matrix* first defined by Bird (1976) in the context of minimum cost spanning tree problems. Even though the path depends on t, it is possible to show that $\overline{c}(r,s)$ is independent of the chosen t (see Aarts and Driessen (1993)).

Let $\overline{\mathcal{P}} = \bigcup_{P_r \in \mathcal{P}} P_r$. Given any $i \in \overline{\mathcal{P}}$, let $P^i \in \mathcal{P}$ such that $i \in P^i$.

Given any $t \in MT^{\mathcal{P}}$ and $i, j \in \overline{\mathcal{P}}$, let $\overline{c}(i,j)$ denote the maximum cost in the (unique) path between P^i and P^j in t. Formally, given $\{r_0, ..., r_k\}$ be the (unique) path between $P^i = r_0$ and $P^j = r_k$ in t, we define $\overline{c}(i,j) = \overline{c}(r_0, r_k)$.

Notice that this definition implies $\overline{c}(i,j) = 0$ when $P^i = P^j$.

Again, $\overline{c}(i,j)$ is independent of the chosen t.

Let $\Pi_0^{\overline{\mathcal{P}}}$ denote the set of orderings of agents in $\overline{\mathcal{P}}$ with 0 as first element. Namely,

$$\Pi_0^{\overline{\mathcal{P}}} = \left\{\pi : \left\{1, \ldots, \left|\overline{\mathcal{P}}\right|\right\} \longrightarrow \overline{\mathcal{P}} : \pi \text{ biyective and } \pi(1) = 0\right\}.$$

Given $\pi \in \Pi_0^{\overline{\mathcal{P}}}$, we define $\overline{\Psi}^{\pi}(\mathcal{P}, c) \in \mathbb{R}^{\overline{\mathcal{P}} \setminus \{0\}}$ inductively as follows:

$$\overline{\Psi}^{\pi}_{\pi(l)}(\mathcal{P}, c) = \min_{l'=1,\ldots,l-1} \overline{c}\left(\pi\left(l'\right), \pi\left(l\right)\right)$$

for all $l = 2, \ldots, |\mathcal{P}|$.

We define $\Psi(\mathcal{P}, c)$ as follows. Given $r \in \mathcal{P} \setminus \{0\}$,

$$\Psi_r(\mathcal{P}, c) = \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \sum_{\pi \in \Pi_0^{\overline{\mathcal{P}}}} \sum_{j \in P_r} \overline{\Psi}^{\pi}_j(\mathcal{P}, c).$$

We now derive a simplified formula for Ψ. Let r^{π} denote the first agent in P_r for the order π. Then, $\overline{\Psi}^{\pi}_i(\mathcal{P}, c) = 0$ for all $i \in P_r \setminus \{r^{\pi}\}$. Analogously, $\overline{\Psi}^{\pi}_i(\mathcal{P}, c) = 0$ for all $i \in P_0 \setminus \{0\}$. Hence, $\sum_{j \in P_r} \overline{\Psi}^{\pi}_j(\mathcal{P}, c) = \overline{\Psi}^{\pi}_{r^{\pi}}(\mathcal{P}, c)$. Let $\Psi^{\pi}(\mathcal{P}, c) \in \mathbb{R}^{\mathcal{P} \setminus \{0\}}$ be defined as

$$\Psi^{\pi}_r(\mathcal{P}, c) = \overline{\Psi}^{\pi}_{r^{\pi}}(\mathcal{P}, c)$$

for all $r \in \mathcal{P}$. From this, the definition of Ψ reduces to:

$$\Psi(\mathcal{P}, c) = \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \sum_{\pi \in \Pi_0^{\overline{\mathcal{P}}}} \Psi^{\pi}(\mathcal{P}, c).$$

Example 4.1 *Let $\mathcal{P} = \{0, r, s\}$ with $P_0 = \{0\}$, $P_r = \{1, 2\}$ and $P_s = \{3\}$, and let $c(r, 0) = 15$, $c(s, 0) = 17$, and $c(r, s) = 3$. This situation is depicted in Figure 2.*

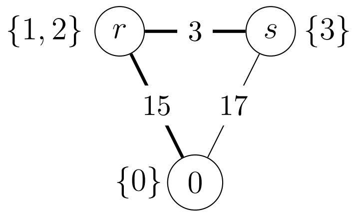

Figure 2: Situation in which agents 1 and 2 act as a single node.

In this situation there exists a unique mt, $t = \{(0, r), (r, s)\}$. Table 4.1 presents each possible $\Psi^\pi(\mathcal{P}, c)$ as well as its average, $\Psi(\mathcal{P}, c) = (11, 7)$.

π	Ψ^π_r	Ψ^π_s
[0123]	15	3
[0132]	15	3
[0213]	15	3
[0231]	15	3
[0312]	3	15
[0321]	3	15
Average	11	7

Table 1: Ψ as an average over orders of the agents.

Main characterization

We next prove that Ψ satisfies all the relevant properties (Theorem 4.1 and Corollary 4.2) and it is characterized by them (Theorem 4.2).

Theorem 4.1 *Ψ satisfies Population Monotonicity, Solidarity, Strong Merge-Proofness, Efficient Merging, Piece-wise Additivity, and Symmetry.*

Proof. We check first that Ψ satisfies Population Monotonicity. Fix $r, s \in \mathcal{P} \setminus \{0\}$. For each $\pi \in \Pi_0^{\overline{\mathcal{P}}}$, let π^{-s} be the order in $\overline{\mathcal{P}} \setminus P_s$ induced by π by removing the agents in coalition s. Moreover, $\overline{\mathcal{P} \setminus \{s\}} = \overline{\mathcal{P}} \setminus P_s$. Now, for each $\varpi \in \Pi_0^{\overline{\mathcal{P}} \setminus P_s}$, we have

$$\left|\left\{\pi \in \Pi_0^{\overline{\mathcal{P}}} : \pi^{-s} = \varpi\right\}\right| = \frac{\left(\left|\overline{\mathcal{P}}\right| - 1\right)!}{\left(\left|\overline{\mathcal{P}}\right| - |P_s| - 1\right)!} = \frac{\left|\Pi_0^{\overline{\mathcal{P}}}\right|}{\left|\Pi_0^{\overline{\mathcal{P} \setminus \{s\}}}\right|}.$$

Let α_s denote this cardinal. Notice that α_s does not depend on the particular ϖ. Moreover, the maximum cost of a path between r and any other node cannot increase in we add new nodes. Hence, for each $\pi \in \Pi_0^{\overline{\mathcal{P}}}$, we have $\Psi_r^\pi(\mathcal{P}, c) \leq \Psi_r^{\pi^{-s}}(P \setminus \{s\}, c)$ and thus

$$\begin{aligned}
\Psi_r(\mathcal{P}, c) &\leq \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \sum_{\pi \in \Pi_0^{\overline{\mathcal{P}}}} \Psi_r^{\pi^{-s}}(P \setminus \{s\}, c) \\
&= \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \sum_{\varpi \in \Pi_0^{\overline{\mathcal{P} \setminus \{s\}}}} \left(\sum_{\pi \in \Pi_0^{\overline{\mathcal{P}}} : \pi^{-s} = \varpi} \Psi_r^{\varpi}(P \setminus \{s\}, c) \right) \\
&= \frac{\alpha_c}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \sum_{\varpi \in \Pi_0^{\overline{\mathcal{P} \setminus \{s\}}}} \Psi_r^{\varpi}(P \setminus \{s\}, c) = \Psi_r(\mathcal{P} \setminus \{s\}, c).
\end{aligned}$$

We check now that Ψ satisfies Solidarity. Notice first that the maximum cost of a path between any pair of nodes cannot decrease when we increase the cost of some link, leaving the rest unaffected. Hence, for each $\pi \in \Pi_0^{\overline{\mathcal{P}}}$, Ψ^π satisfies solidarity. Since Ψ is the average of these Ψ^π, and the orders remain unaffected when a cost increases, we deduce that Ψ also satisfies Solidarity.

We check now that Ψ satisfies Strong Merge-Proofness. Notice that the maximum cost of a path between any pair of nodes cannot increase when some other nodes merge. Hence, for each $\pi \in \Pi_0^{\overline{\mathcal{P}}}$, Ψ^π satisfies Strong Merge-Proofness. Since Ψ is the average of these Ψ^π, and the orders remain unaffected when two or more coalitions merge, we deduce that Ψ also satisfies Strong Merge-Proofness.

We check now that Ψ satisfies Efficient Merging. Let $r, s \in \mathcal{P}$ be one of the closest pairs of coalitions. Let $\mathcal{P}^{rs} = (\mathcal{P} \setminus \{r, s\}) \cup \{r \cup s\}$. Then, for any $\pi \in \Pi_0^{\overline{\mathcal{P}}}$ satisfying that there exists some $i \in P_r$ with $\pi(i) < \pi(j)$ for all $j \in P_s$, we have

$$\begin{aligned}
\Psi_r^\pi(\mathcal{P}, c) &= \Psi_{r\cup s}^\pi(\mathcal{P}^{rs}, c) \\
\Psi_s^\pi(\mathcal{P}, c) &= c(r, s).
\end{aligned}$$

Let Π^r the subset of these orderings. Analogously, let Π^s be the subset of orders $\pi \in \Pi_0^{\overline{\mathcal{P}}}$ satisfying that there exists some $i \in P_s$ with $\pi(i) < \pi(j)$ for all $j \in P_r$. It is clear that $\Pi^r \cap \Pi^s = \emptyset$ and $\Pi_0^{\bar{\mathcal{P}}} = \Pi^r \cup \Pi^s$. Hence,

$$\begin{aligned}
\Psi_r(\mathcal{P}, c) &= \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \sum_{\pi \in \Pi_0^{\overline{\mathcal{P}}}} \Psi_r^\pi(\mathcal{P}, c) \\
&= \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \left(\sum_{\pi \in \Pi^r} \Psi_r^\pi(\mathcal{P}, c) + \sum_{\pi \in \Pi^s} \Psi_r^\pi(\mathcal{P}, c) \right) \\
&= \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \left(\sum_{\pi \in \Pi^r} \Psi_{r\cup s}^\pi(\mathcal{P}^{rs}, c) + \sum_{\pi \in \Pi^s} c(r, s) \right)
\end{aligned}$$

analogously

$$\Psi_s(\mathcal{P}, c) = \frac{1}{\left|\Pi_0^{\overline{\mathcal{P}}}\right|} \left(\sum_{\pi \in \Pi^s} \Psi_{r\cup s}^\pi(\mathcal{P}^{rs}, c) + \sum_{\pi \in \Pi^r} c(r, s) \right)$$

so that, taking into account that $\overline{\mathcal{P}} = \overline{\mathcal{P}^{rs}}$,

$$\Psi_r(\mathcal{P}, c) + \Psi_s(\mathcal{P}, c) = \Psi_{r\cup s}(\mathcal{P}^{rs}, c) + c(r, s).$$

The "greater or equal" part of this equality constitutes the proof of Efficient Merging.

We check now that Ψ satisfies Piece-wise Additivity. Let c, c' be two cost functions on the same cone C_σ. Hence, there exists a common *mt* t in both $(\mathcal{P}, c)$ and $(\mathcal{P}, c')$ and, moreover, t is also a *mt* in $(\mathcal{P}, c + c')$. From this, it follows that for any $\pi \in \Pi_0^{\overline{\mathcal{P}}}$,

$$\Psi_r^\pi(\mathcal{P}, c + c') = \Psi_r^\pi(\mathcal{P}, c) + \Psi_r^\pi(\mathcal{P}, c').$$

Since Ψ is the average of these Ψ^{π}, we deduce that Ψ satisfies Piece-wise Additivity.

Finally, the proof that Ψ satisfies Symmetry follows from its definition. ■

A relevant implication of this result is that Ψ coincides with the folk solution in minimum cost spanning tree problems:

Proposition 4.1 *Let* $(\mathcal{P}, c)$ *be such that* $|P_r| = 1$ *for all* $r \in \mathcal{P}$. *Then,* $\Psi(\mathcal{P}, c)$ *coincides with the folk rule of the minimum cost spanning tree problem* $\left(\overline{\mathcal{P}}, c\right)$.

Proof. It follows from Theorem 1 in Bergantiños and Vidal-Puga (2009) that the folk rule is the only one that satisfies Piece-wise Additivity, Symmetry and Separability, which is a weaker property than Population Monotonicity. From Theorem 4.1, Ψ satisfies these properties and hence it coincides with the folk rule when $|P_r| = |P_s|$ for all $r, s \in \mathcal{P} \setminus \{0\}$. ■

From the proof of Proposition 4.1, it is clear that Ψ coincides with the folk rule when all the nodes contain the same number of agents. The result does not longer apply when these cardinalities are different, because in that case the nodes cannot be symmetric.

Notice that Ψ satisfies a stronger version of Piece-wise Additivity. In particular, Ψ is piece-wise linear as a function with domain $\mathbb{R}^{E^{\mathcal{P}}}$. This allow us to characterize Ψ by providing its value for any c with $c(i, j) \in \{0, 1\}$ for all i, j. We do it now. Let c^{01} be such a cost function.

For any $(\mathcal{P}, c^{01})$, the total cost is $m(\mathcal{P}, c^{01}) = |\mathbb{P}| - 1$, where $\mathbb{P}$ is the partition of $\mathcal{P}$ in connected components. For each $R \in \mathbb{P}$, let $\overline{R} = \bigcup_{r \in R} P_r$ be the set of agents that form R. It is then straightforward to check that Ψ is characterized by:

$$\Psi_r\left(\mathcal{P}, c^{01}\right) = \begin{cases} 0 & \text{if } r \in S_0 \\ \frac{|P_r|}{\left|\overline{R}\right|} & \text{if } r \in R \in \mathbb{P} \setminus \{S_0\} \end{cases} \tag{1}$$

for all $r \in \mathcal{P} \setminus \{0\}$.

We now provide a necessary condition for Merge-Proofness and Efficient Merging.

Proposition 4.2 *If Φ satisfies Strong Merge-Proofness and Efficient Merging, then*

$$\Phi_r(\mathcal{P}, c) = \sum_{i \in P_r} \Phi_{\{i\}}\left(\mathcal{P}^*, c^*\right)$$

for all $r \in \mathcal{P}$, where $\mathcal{P}^* = \{\{i\}\}_{i \in \overline{\mathcal{P}}}$ and c^* is defined as $c^*\left(\{i\}, \{j\}\right) = c(P^i, P^j)$ for all $i, j \in \overline{\mathcal{P}}$.

Proof. Let Φ be a rule that satisfies Strong Merge-Proofness and Efficient Merging. We proceed by induction on $|\mathcal{P}|$. For $|\mathcal{P}| = 2$, we have $\mathcal{P} = \{r, 0\}$ and $c^*(\{i\}, \{j\}) = 0$ for all $i, j \in P_r$. Thus,

$$\Phi_r(\mathcal{P}, c) = c(r, 0) = m(\mathcal{P}^*, c^*) = \sum_{i \in P_r} \Phi_{\{i\}}(\mathcal{P}^*, c^*).$$

Assume now $|\mathcal{P}| > 2$ and the result holds when there are less than $|\mathcal{P}|$ coalitions. The result also holds trivially when $|P_r| = 1$ for all $r \in \mathcal{P}$. Hence, assume there exists some $r \in \mathcal{P}$ with $|P_r| > 1$. Since $c^*(\{i\}, \{j\}) = 0$ for all $i, j \in P_r$, we can apply Efficient Merging sequentially on each $i \in P_r$ to obtain

$$\sum_{i \in P_r} \Phi_{\{i\}}(\mathcal{P}^*, c^*) \geq \Phi_r(\mathcal{P}^{*r}, c^*) \tag{2}$$

where $\mathcal{P}^{*r} = (\mathcal{P}^* \setminus \{\{i\}\}_{i \in P_r}) \cup \{r\}$. Now,

$$\begin{aligned} \Phi_r(\mathcal{P}^{*r}, c^*) &= m(\mathcal{P}^{*r}, c^*) - \sum_{j \in \overline{\mathcal{P}} \setminus P_r} \Phi_{\{j\}}(\mathcal{P}^{*r}, c^*) \\ &\geq m(\mathcal{P}^{*r}, c^*) - \sum_{j \in \overline{\mathcal{P}} \setminus P_r} \Phi_{\{j\}}(\mathcal{P}^*, c^*) \\ &= m(\mathcal{P}^*, c^*) - \sum_{j \in \overline{\mathcal{P}} \setminus P_r} \Phi_{\{j\}}(\mathcal{P}^*, c^*) = \sum_{i \in P_r} \Phi_{\{i\}}(\mathcal{P}^*, c^*) \qquad (3) \end{aligned}$$

where the second inequality comes from Strong Merge-Proofness. Combining (2) and (3), we see that all weak inequalities are equalities, and in particular:

$$\sum_{i \in P_r} \Phi_{\{i\}}(\mathcal{P}^*, c^*) = \Phi_r(\mathcal{P}^{*r}, c^*).$$

■

Corollary 4.1 *Solidarity, Merge-Proofness, Efficient Merging and Symmetry imply Positivity.*

Proof. Let Φ be a rule satisfying these properties. By Proposition 4.2, $\Phi_r(\mathcal{P}, c) = \sum_{i \in P_r} \Phi_{\{i\}}(\mathcal{P}^*, c^*)$. Under Solidarity, $\Phi(\mathcal{P}^*, c^*) \geq \Phi(\mathcal{P}^*, c^0)$ where $c^0(i,j) = 0$ for all i, j. Under Symmetry, $\Phi(\mathcal{P}^*, c^0) = (0, \ldots, 0)$ and hence $\Phi(\mathcal{P}, c) \geq (0, \ldots, 0)$. ■

Corollary 4.2 Ψ *satisfies Core Selection, Cost Monotonicity, Merge-Proofness, Split-Proofness, and Positivity.*

Proof. It follows from Proposition 3.1, Theorem 4.1, Corollary 4.1, and the fact that Population Monotonicity implies Core Selection, Solidarity implies Cost Monotonicity, and Strong Merge-Proofness implies Merge-Proofness. ■

We now present our main result:

Theorem 4.2 Ψ *is the only rule that satisfies Population Monotonicity, Solidarity, Strong Merge-Proofness, Efficient Merging, Piece-wise Additivity, and Symmetry.*

Proof. We already know that Ψ satisfies all these properties. Let Φ be a rule that satisfies them. Under Corollary 4.1, Φ also satisfies Positivity. We will prove that Φ is unique for each $(\mathcal{P}, c)$. We proceed by induction on the number coalitions in $\mathcal{P}$. If $|\mathcal{P}| = 1$, the result is trivial. Assume then that the result is true when there are less than $|\mathcal{P}|$ coalitions.

Under Strong Merge-proofness and Efficient Merging, by Proposition 4.2 it is enough to prove the result assuming $|P_r| = 1$ for all $r \in \mathcal{P}$.

Under Piece-wise Additivity, it is enough to prove the result assuming that c only takes two values: 0 and some $x \in \mathbb{R}_+$. To see why, notice that every $(\mathcal{P}, c)$ can be expressed as the sum of these situations, all of them in the same cone C_σ for some σ satisfying $c(\sigma(l)) \leq c(\sigma(l'))$ iff $l \leq l'$.

Under Population Monotonicity and the induction hypothesis, we can assume that there exists a spanning graph g in $\mathcal{P} \setminus \{P_0\}$ such that $c(r, s) = 0$

for all $(r, s) \in g$. Suppose, on the contrary, that there exist two groups of coalitions $\mathcal{Q}, \mathcal{Q}' \subset \mathcal{P}$ such that $\mathcal{Q} \cup \mathcal{Q}' = \mathcal{P}$ and $c(r, r') = x$ for all $r \in \mathcal{Q}$ and $r' \in \mathcal{Q}'$. Then Population Monotonicity implies that $\Phi_r(\mathcal{Q}, c) \geq \Phi_r(\mathcal{P}, c)$ for all $r \in \mathcal{Q}$ and $\Phi_r(\mathcal{Q}', c) \geq \Phi_r(\mathcal{P}, c)$ for all $r \in \mathcal{Q}'$. Moreover, it is straightforward to check that $m(\mathcal{P}, c) = m(\mathcal{Q}, c) + m(\mathcal{Q}', c)$. Hence, given $r \in \mathcal{Q}$ (the case $r \in \mathcal{Q}'$ is analogous),

$$\begin{aligned}\Phi_r(\mathcal{P}, c) &= m(\mathcal{P}, c) - \sum_{s \in \mathcal{Q} \setminus \{r\}} \Phi_s(\mathcal{P}, c) - \sum_{s \in \mathcal{Q}'} \Phi_s(\mathcal{P}, c) \\ &\geq m(\mathcal{Q}, c) + m(\mathcal{Q}', c) - \sum_{s \in \mathcal{Q} \setminus \{r\}} \Phi_s(\mathcal{Q}, c) - \sum_{s \in \mathcal{Q}'} \Phi_s(\mathcal{Q}', c) \\ &= \Phi_r(\mathcal{Q}, c) \geq \Phi_r(\mathcal{P}, c)\end{aligned}$$

and so $\Phi_r(\mathcal{P}, c) = \Phi_r(\mathcal{Q}, c)$, which is unique by induction hypothesis.

Under Positivity, it is enough to prove the result assuming that $c(r, 0) = x$ for all $r \in \mathcal{P} \setminus \{0\}$. Suppose, on the contrary, that $c(r, 0) = 0$ for some $r \in \mathcal{P} \setminus \{0\}$. Hence, $m(\mathcal{P}, c) = 0$ because $g \cup \{(r, 0)\}$ is a spanning graph with cost 0. Under Positivity, $\Phi(\mathcal{P}, c) \geq (0, \ldots, 0)$ but since $\sum_{s \in \mathcal{P} \setminus \{0\}} \Phi_s(\mathcal{P}, c) = m(\mathcal{P}, c) = 0$, we conclude that $\Phi(\mathcal{P}, c) = (0, \ldots, 0)$.

Clearly, under these assumptions we have $m(\mathcal{P}, c) = x$. Assume w.l.o.g. $\overline{\mathcal{P}} = \{0, 1, \ldots, p\}$. In particular, this implies $\mathcal{P} = \{\{0\}, \{1\}, \ldots, \{p\}\}$. Let c^{0x} be the cost function defined as $c^{0x}(r, 0) = x$ for all $r \in \mathcal{P} \setminus \{0\}$ and $c^{0x}(r, s) = 0$ otherwise. For each $i \in \{1, \ldots, p\}$, let $P_r \subset \overline{\mathcal{P}}$ be the set of nodes whose (unique) path to the source in g uses node i (including node i itself), and let $P_s = \left(\overline{\mathcal{P}} \setminus P_r\right) \setminus \{0\}$. Both sets P_r and P_s can be connected at zero cost. Hence, under Efficient Merging and Strong Merge-Proofness, we have

$$\sum_{j \in P_r} \Phi_{\{j\}}(\mathcal{P}, c) = \Phi_r(\{0, r, s\}, c) = \Phi_r\left(\{0, r, s\}, c^{0x}\right) = \sum_{j \in P_r} \Phi_{\{j\}}\left(\mathcal{P}, c^{0x}\right).$$

Under Symmetry, $\Phi_{\{j\}}\left(\mathcal{P}, c^{0x}\right) = \frac{x}{p}$ for all $j \in \overline{\mathcal{P}}$. Hence, $\sum_{j \in P_r} \Phi_{\{j\}}(\mathcal{P}, c) = \frac{x|P_r|}{p}$. We can now proceed by induction on $|P_r|$ in order to prove that $\Phi(\mathcal{P}, c) = \left(\frac{x}{p}, \ldots, \frac{x}{p}\right)$. For $|P_r| = 1$, we have $P_r = \{i\}$ and hence $\Phi_{\{i\}}(\mathcal{P}, c) =$

$\frac{x}{p}$. Assume now $\Phi_{\{i\}}(\mathcal{P},c)=\frac{x}{p}$ when $|P_r|<\alpha$ and suppose $|P_r|=\alpha$. Then,

$$\Phi_{\{i\}}(\mathcal{P},c)=\sum_{j\in P_r}\Phi_{\{j\}}(\mathcal{P},c)-\sum_{j\in P_r\setminus\{i\}}\Phi_{\{j\}}(\mathcal{P},c)=\frac{x|P_r|}{p}-\sum_{j\in P_r\setminus\{i\}}\frac{x}{p}=\frac{x}{p}.$$

■

Notice, from the proof of Theorem 4.2, that we can replace Solidarity by Positivity in the characterization result. In either case, the properties are independent, as we show in the next subsection.

Independence of the properties

We present six reasonable rules that satisfy the properties used in Theorem 4.2 but one.

Without Piece-wise Additivity: Let F^e be defined as

$$F_r^e(\mathcal{P},c)=\sum_{i\in P_r}\left(\overline{c}(i,0)-\sum_{S\ni i:0\notin S\subset N,\delta_S>0}(1-e_i(\overline{c},S))\,\delta_S\right)$$

where $\delta_S=\min_{i\in S,j\in N_0\setminus S}\overline{c}(i,j)-\max_{i,j\in S}\overline{c}(i,j)$ determines the extra-cost that agents in S should face after they get connected, and e is normalized extra-cost function that assigns to each irreducible cost function $\overline{c}$ a vector in the simplex Δ^S. See Bergantiños and Vidal-Puga (2015) for a detailed interpretation of these terms. Let $M(\overline{c},S)=\{i\in S:\overline{c}(i,j)\leq\overline{c}(k,j)\text{ for all }j,k\in S\}$ be the set of agents that are closer under $\overline{c}$ within S. When e is defined as

$$e_i(\overline{c},S)=\begin{cases}\frac{1}{|S|+1}+\frac{1}{(|S|+1)(|M(\overline{c},S)|)} & \text{if } i\in M(\overline{c},S)\\ \frac{1}{|S|+1} & \text{otherwise}\end{cases}$$

then F^e is a cost-sharing rule that satisfies all the properties but Piece-wise Additivity. In Example 4.1, $F^e(\mathcal{P},c)=(12,6)$.

The rest of the rules are piece-wise linear, and hence it is enough to define them for any c^{01} with $c^{01}(i,j)\in\{0,1\}$ for all i,j (as in (1)).

For any $(\mathcal{P}, c^{01})$ and $r \in \mathcal{P} \setminus \{0\}$, let $\Lambda_r \subset \overline{\mathcal{P}}$ be the set of agents that belong to either r or to some s such that there exists a path between r and s with zero cost. Let $\lambda_r = |\{s \in \Lambda_r : c^{01}(s,0) = 0\}|$ be the number of nodes in Λ_r with zero cost to the source. It is not difficult to check that $P_r \subset S_0$ if and only if $\lambda_r > 0$.

Without Symmetry: Given $\pi \in \Pi^N$, we consider Ψ^π. In Example 4.1, if $\pi(i) = i$ for all $i \in N$, we have $\Psi^\pi(\mathcal{P}, c) = (15, 3)$.

Without Population Monotonicity: Given $\alpha \in (0,1)$, let Φ^α be defined by

$$\Phi_r^\alpha\left(\mathcal{P}, c^{01}\right) = \begin{cases} 0 & \text{if } S_0 = \mathcal{P} \\ \frac{|P_r|\alpha}{\sum_{P_s \in R \setminus \{0\}} |P_s|} & \text{if } R = S_0 \neq \mathcal{P} \\ \frac{|P_r|}{\sum_{P_s \in R} |P_s|} & \text{if } R \neq S_0 = \{0\} \\ \frac{|P_r|}{\sum_{P_s \in R} |P_s|} - \frac{|P_r|\alpha}{\sum_{P_s \in \mathcal{P} \setminus S_0} |P_s|} & \text{if } R \neq S_0 \neq \{0\} \end{cases}$$

for all $r \in \mathcal{P} \setminus \{0\}$, where $R \in \mathbb{P}$ is such that $P_r \in R$. This Φ^α is a subsidy rule where the cost of connection of the nodes that are far away from the source in term of costs (those in $\mathcal{P} \setminus S_0$) are partially subsidized by the nodes that are closer. In Example 4.1, $\Phi^\alpha(\mathcal{P}, c) = (11, 7)$. Notice that it coincides with $\Psi(\mathcal{P}, c)$. In this example, there are no nodes far away from the source (with respect to the others). Assume now that there exists a third node $P_u = \{4\}$ with $c(0,u) = c(r,u) = c(s,u) = 20$ for all $i \in \{0,1,2,3\}$. Then, $\Phi^\alpha(\mathcal{P}, c) = \left(11 + \frac{10\alpha}{3}, 7 + \frac{5\alpha}{3}, 20 - 5\alpha\right)$ whereas $\Psi(\mathcal{P}, c) = (11, 7, 20)$.

Without Solidarity: Let Φ^λ be defined by

$$\Phi_r^\lambda\left(\mathcal{P}, c^{01}\right) = \begin{cases} \frac{|P_r|}{\sum_{s \in \mathcal{P}: P_s \subset \Lambda_r} |P_s|} & \text{if } \lambda_r = 0 \\ \frac{|P_r|}{\sum_{s \in \mathcal{P}: P_s \subset \Lambda_r} |P_s| - 1} & \text{if } \lambda_r = 1 \text{ and } c^{01}(r,0) = 1 \\ \frac{|P_r| - 1}{\sum_{s \in \mathcal{P}: P_s \subset \Lambda_r} |P_s| - 1} - 1 & \text{if } \lambda_r = 1 \text{ and } c^{01}(r,0) = 0 \\ 0 & \text{if } \lambda_r > 1 \end{cases}$$

for all $r \in \mathcal{P} \setminus \{0\}$. When there are coalitions that connect to the source through a unique node, they should pay a compensation to this node. In Example 4.1, $\Phi^{\lambda}(\mathcal{P}, c) = (10, 8)$.

Without Efficient Merging: Let Φ^2 be defined by

$$\Phi_r^2\left(\mathcal{P}, c^{01}\right) = \begin{cases} 0 & \text{if } P_r \subset S_0 \\ \frac{|P_r|^2}{\sum_{s \in \mathcal{P}: P_s \subset \Lambda_r} |P_s|^2} & \text{if } P_r \subset R \in \mathbb{P} \setminus \{S_0\} \end{cases}$$

for all $r \in \mathcal{P} \setminus \{0\}$. In Example 4.1, $\Phi^2(\mathcal{P}, c) = (12.6, 5.4)$.

Without Strong Merge-Proofness: Let $\tilde{\Phi}$ be defined by

$$\tilde{\Phi}_r\left(\mathcal{P}, c^{01}\right) = \begin{cases} 0 & \text{if } P_r \subset S_0 \\ \frac{1}{|\{s \in \mathcal{P}: P_s \subset R \in \mathbb{P} \setminus \{S_0\}\}|} & \text{if } P_r \subset R \in \mathbb{P} \setminus \{S_0\} \end{cases}$$

for all $r \in \mathcal{P} \setminus \{0\}$. This rule is similar to Ψ, but averaging on the different orders of coalitions, instead of agents. It is equivalent to the folk rule when the size of the nodes is not taken into account. In Example 4.1, $\tilde{\Phi}(\mathcal{P}, c) = (9, 9)$.

5 The rule as a weighted Shapley value

Another remarkable implication of the characterization result is that Ψ can be defined as a the weighted Shapley value of an appropriate transfer utility (TU) cost game.

For each $(\mathcal{P}, c)$, consider the TU cost game $(\mathcal{P} \setminus \{0\}, v^+)$ defined as $v^+(R) = m\left(R \cup \{0\}, c^R\right)$ for all $R \subset \mathcal{P} \setminus \{0\}$, where $c^R(r, s) = c(r, s)$ for all $r, s \in R$, and $c^R(r, 0) = \min_{s \in \mathcal{P} \setminus R} c(r, s)$ for all $r \in R$.

This TU cost game was first defined by Bergantiños and Vidal-Puga (2007b). It follows an "optimistic" interpretation of the worth of a coalition of players, since it assumes that the rest of the players are already connected and it is possible to get to the source through them.

Example 5.1 *Consider $(\mathcal{P}, c)$ defined in Example 4.1. The TU cost game $(\{r, s\}, v^+)$ is given by $v^+(\{r\}) = v^+(\{s\}) = 3$ and $v^+(\{r, s\}) = 18$.*

We now interpret Ψ as a weighted Shapley value of this TU cost game:

Proposition 5.1 *For any* $(\mathcal{P}, c)$*,* $\Psi(\mathcal{P}, c)$ *is the weighted Shapley value of the TU cost game* $(\mathcal{P} \setminus \{0\}, v^+)$ *where the weights are given by* $\omega_r = |P_r|$ *for each* $r \in \mathcal{P} \setminus \{0\}$*.*

Proof. Let c^{01} be a cost function with $c(i,j) \in \{0,1\}$ for all i,j. Since the weighted Shapley value is linear in the characteristic function, it is enough to prove that it coincides with $\Psi(\mathcal{P}, c^{01})$ for each such c^{01}. It is clear that the TU cost game $(\mathcal{P} \setminus \{0\}, v^+)$ associated with c^{01} is given by

$$v^+ = \sum_{R \in \mathbb{P} \setminus \{S_0\}} u_R$$

where u_R is the unanimity game with carrier R, i.e. $u_R(S) = 1$ if $R \subset S$ and $u_R(S) = 0$ otherwise. Moreover, by definition of the weighted Shapley value (as defined by Kalai and Samet (1987)),

$$Sh_r^\omega(\mathcal{P} \setminus \{0\}, u_R) = \begin{cases} 0 & \text{if } r \notin R \\ \frac{\omega_r}{\sum_{s \in R} \omega_s} & \text{if } r \in R \end{cases}$$

for all $r \in \mathcal{P} \setminus \{0\}$. Since $\mathbb{P}$ is a partition of $\mathcal{P}$, by additivity of Sh^ω,

$$Sh_r^\omega\left(\mathcal{P} \setminus \{0\}, v^+\right) = \begin{cases} 0 & \text{if } r \notin \bigcup_{R \in \mathbb{P} \setminus \{S_0\}} R \\ \frac{\omega_r}{\sum_{s \in R} \omega_s} & \text{if } r \in R \in \mathbb{P} \setminus \{S_0\} \end{cases} \tag{4}$$

for all $r \in \mathcal{P} \setminus \{0\}$. It is clear that $r \notin \bigcup_{R \in \mathbb{P} \setminus \{S_0\}} R$ iff $r \in S_0$ and, by definition, $\omega_r = |P_r|$ and $\sum_{s \in R} \omega_s = \sum_{s \in R} |P_s| = \left|\overline{R}\right|$. Hence, the right-part of (1) and (4) coincide, so $\Psi(\mathcal{P}, c^{01}) = Sh^\omega(\mathcal{P} \setminus \{0\}, v^+)$. ■

Bergantiños and Vidal-Puga (2007b) proved that the folk rule coincides with the Shapley value of the optimistic TU cost game. From this, Proposition 4.1 can also be derived from Proposition 5.1.

6 Conclusions

In the classical model of minimum cost spanning tree problems, it is assumed that the planner is not able to distinguish who are the agents that belong to each node. In many situations, such as those in which the agents are represented by geographical points, it seems reasonable that the planner can identify how many agents may belong to the same node. This assumption is very reasonable in the particular case of minimum cost spanning tree problems, where it is common knowledge that all the nodes want to be connected to the source. Given this, we study three classes of properties for a rule to satisfy. These classes are related to core selection, cost monotonicity and split and merge-proofness, respectively. While in the classical model there is no rule satisfying merge-proofness, here we propose one satisfying the three classes of properties. We also provide a characterization using symmetry, piece-wise additivity and several variations of core selection, cost monotonicity and split and merge-proofness. Symmetry is an important property from the point of view of equity. On the other hand, we do not claim that piece-wise additivity is an essential property, but it provides a linear structure to the solution and, as such, it allows us to pick up a single reasonable rule.

References

Aarts, H. and Driessen, T. (1993). The irreducible core of a minimum cost spanning tree game. *Methods and Models of Operations Research*, 38:163–174.

Athanassoglou, S. and Sethuraman, J. (2008). A characterization result in minimum cost spanning tree games. *Mimeo.*

Bergantiños, G. and Gómez-Rúa, M. (2010). Minimum cost spanning tree problems with groups. *Economic Theory*, 43:227–262.

Bergantiños, G., Lorenzo, L., and Lorenzo-Freire, S. (2010). The family of cost monotonic and cost additive rules in minimum cost spanning tree problems. *Social Choice and Welfare*, 34:695–710.

Bergantiños, G. and Vidal-Puga, J. (2007a). A fair rule in minimum cost spanning tree problems. *Journal of Economic Theory*, 137:326–352.

Bergantiños, G. and Vidal-Puga, J. (2007b). The optimistic TU game in minimum cost spanning tree problems. *International Journal of Game Theory*, 36:223–239.

Bergantiños, G. and Vidal-Puga, J. (2008). On some properties of cost allocation rules in minimum cost spanning tree problems. *AUCO Czech Economic Review*, 2(3):251–267.

Bergantiños, G. and Vidal-Puga, J. (2009). Additivity in minimum cost spanning tree problems. *Journal of Mathematical Economics*, 1-2:38–42.

Bergantiños, G. and Vidal-Puga, J. (2015). Characterization of monotonic rules in minimum cost spanning tree problems. *International Journal of Game Theory*, Forthcoming.

Bird, C. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks*, 6:335–350.

Bogomolnaia, A. and Moulin, H. (2010). Sharing the cost of a minimal cost spanning tree: beyond the folk solution. *Games and Economic Behavior*, 69(2):238–248.

Branzei, R., Moretti, S., Norde, H., and Tijs, S. (2004). The P-value for cost sharing in minimum cost spanning tree situations. *Theory and Decision*, 56:47–61.

Chun, Y. (1988). The proportional solution for right problems. *Mathematical Social Sciences*, 15:231–246.

De Frutos, M. (1999). Coalitional manipulation in a bankruptcy problem. *Review of Economic Design*, 4:255–272.

Dutta, B. and Kar, A. (2004). Cost monotonicity, consistency and minimum cost spanning tree games. *Games and Economic Behavior*, 48(2):223–248.

Feltkamp, V., Tijs, S., and Muto, S. (1994). On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems. *CentER DP 1994 nr.106, Tilburg University, The Netherlands.*

Gómez-Rúa, M. and Vidal-Puga, J. (2011). Merge-proofness in minimum cost spanning tree problems. *International Journal of Game Theory*, 40(2):309–329.

Granot, D. and Huberman, G. (1981). On minimum cost spanning tree games. *Mathematical Programming*, 21:1–18.

Granot, D. and Huberman, G. (1984). On the core and nucleolus of minimum cost spanning tree problems. *Mathematical Programming*, 29:323–347.

Hart, S. (1974). Formation of cartels in large markets. *Journal of Economic Theory*, 7:453–466.

Hougaard, J. L., Moulin, H., and Østerdal, L. P. (2010). Decentralized pricing in minimum cost spanning trees. *Economic Theory*, 44:293–306.

Ju, B.-G. (2003). Manipulation via merging and splitting in claims problems. *Review of Economic Design*, 8:205–215.

Ju, B.-G., Miyagawa, E., and Sakai, T. (2007). Non-manipulable division rules in claim problems and generalizations. *Journal of Economic Theory*, 132:1–26.

Kalai, E. and Samet, D. (1987). On weighted Shapley values. *International Journal of Game Theory*, 16:205–222.

Kar, A. (2002). Axiomatization of the Shapley value on minimum cost spanning tree games. *Games and Economic Behavior*, 38:265–277.

Knudsen, P. H. and Østerdal, L. P. (2012). Merging and splitting in cooperative games: some (im)possibility results. *International Journal of Game Theory*, 41(4):763–774.

Kruskal, J. (1956). On the shortest spanning subtree of a graph and the travelling salesman problem. *Proceedings of the American Mathematical Society*, 7:48–50.

Legros, P. (1987). Disadvantageous syndicates and stable cartels: the case of the nucleolus. *Journal of Economic Theory*, 42:30–49.

Lorenzo, L. and Lorenzo-Freire, S. (2009). A characterization of Kruskal sharing rules for minimum cost spanning tree problems. *International Journal of Game Theory*, 38:107–126.

Maschler, M. (1976). An advantage of the bargaining set over the core. *Journal of Economic Theory*, 13:184–192.

Moulin, H. (2002). Axiomatic cost and surplus sharing. In Arrow, K., Sen, A., and Suzumura, K., editors, *Handbook of Social Choice and Welfare*, pages 289–357. Elsevier, Amsterdam.

Moulin, H. (2007). On scheduling fees to prevent merging, splitting and transferring of jobs. *Mathematics of Operations Research*, 32(2):266–283.

Moulin, H. (2008). Proportional scheduling, split-proofness, and merge-proofness. *Games and Economic Behavior*, 63:567–587.

Moulin, H. (2014). Pricing traffic in a spanning network. *Games and Economic Behavior*, 86:475–490.

Norde, H., Moretti, S., and Tijs, S. (2004). Minimum cost spanning tree games and population monotonic allocation schemes. *European Journal of Operational Research*, 154(1):84–97.

O'Neill, B. (1982). A problem of rights arbitration from the talmud. *Mathematical Social Sciences*, 2:345–371.

Özsoy, H. (2006). A characterization of Bird's rule. *Working paper at Rice University*.

Postlewaite, A. and Rosenthal, R. (1974). Disadvantageous syndicates. *Journal of Economic Theory*, 9:324–326.

Prim, R. (1957). Shortest connection network and some generalization. *Bell Systems Technology Journal*, 36:1389–1401.

Rosenmüller, J. and Sudhölter, P. (2004). Cartels via the modiclus. *Discrete Applied Mathematics*, 134:263–302.

Shapley, L. (1953). A value for n-person games. In Kuhn and Tucker, editors, *Contributions to the theory of games II*, pages 307–317. Princeton University Press, Princeton NJ.

Sprumont, Y. (2005). On the discrete version of the Aumann-Shapley cost-sharing method. *Econometrica*, 73(5):1693–1712.

Tijs, S., Moretti, S., Branzei, R., and Norde, H. (2006). The Bird core for minimum cost spanning tree problems revisited: Monotonicity and additivity aspects. In Seeger, A., editor, *Recent Advances in Optimization*, pages 305–322. Springer Berlin Heidelberg.

Trudeau, C. (2012). A new stable and more responsible cost sharing solution for mcst problems. *Games and Economic Behavior*, 75(1):402–412.