Mixed rules in multi-issue allocation situations

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Abstract

Multi-issue allocation situations concentrate on dividing an estate among a group of agents. Each agent's claim is a vector specifying the amount claimed by each agent on each issue. We present a two-stage rule dividing first the estate among the issues following the constrained equal awards rule. In the second stage the amount assigned to each issue is proportionally divided among the agents according to their demands on this issue.

Keywords: Bankruptcy; Multi-issue allocation situations; Proportional rule; Constrained equal awards rule.

JEL classification: C79; D63.

1 Introduction

Bankruptcy situations study problems where an estate must be divided among several claimants. The problem arises in the insufficient amount of resources available, more specifically, when the estate is not large enough to cover all claims. The reader is referred to Thomson (2003) for further insight on the literature. In bankruptcy situations each agent's claim is identified by a number. In many real situations we divide an estate among a group of agents, as in bankruptcy for instance, but the claim of each agent is a vector. The government of Spain divides the budget among several issues (health, education, ...) and each "Comunidad Autónoma" (Galicia, Madrid, Catalonia, ...) has claims over the different issues. The government of Galicia divides the budget among several issues (roads, education, ...) and each city council, "Concello", (Vigo, Santiago de Compostela, ...) has claims over the different issues. This kind of situations are called multi-issue allocation (MIA) situations and were introduced in Calleja et al. (2005). They are modeled as a 4-tuple $(R, N, E, (c_{ki})_{k\in R, i\in N})$ where R is the set of

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issues (health, education, ...). N is the set of agents (Galicia, Madrid, Catalonia, ...). E is the estate (the amount to be divided). c_{ki} is the claim of agent i on issue k.

A rule in MIA situations can be defined following two different approaches. In the first one, as in bankruptcy, a rule is a vector which represents the amount assigned, each agent receives. Thus, it is assumed that each agent can divide the total amount among the issues as he pleases. This approach is followed in Calleja et al. (2005), González-Alcón et al. (2007), and Ju et al. (2007). In the second approach a rule is a vector representing the amount assigned each agent receives for each issue. Thus, the amount that each agent receives for each issue must be spent on this issue. The latter approach is more popular in many situations as for instance, in all the situations mentioned above. This approach is followed in Bergantiños et al. (2009, 2010), Lorenzo-Freire et al. (2009), and Moreno-Ternero (2009).

In many Spanish universities, when deciding the annual operating budget that each department will receive, the procedure is as follows. The university decides the money that will be assigned to the departments. The departments submit a quantified amount for each issue, typically research and teaching. Finally, the university decides the amount each department will receive for each issue.

The university authorities argue that research and teaching are two of the University's most significant issues and no one is more important than the other. In our model this idea can be applied by claiming that the amount devoted to research and teaching should be as equal as possible. Nevertheless, within each issue (research or teaching) the authorities argue that agents are, in general, different. Then, the amount they receive should take into account these differences. For instance, in the university of Vigo, the amount each department receives for research is proportional to the number of credit points the department obtains. The credit points are computed taking into account publications, seminars, meetings, books, etc. In our model this idea could be applied by claiming that the amount each agent receives in each issue is proportional to his claim on the issue.

We consider a two-stage rule which divides among the issues according to the Constrained Equal Awards bankruptcy rule and inside each issue following the proportional bankruptcy rule. We give two axiomatic characterizations of this rule. Both are obtained by combining characterizations of both bankruptcy rules.

The paper is organized as follows. In Section 2 we introduce MIA situations. In Section 3 we present our results.

2 Multi-issue allocation situations

A *bankruptcy situation* is a triple (N, E, c). $N = \{1, \ldots, n\}$ is the set of agents, $E \geq 0$ represents the amount to be divided among the agents, and $c = (c_i)_{i \in N} \in \mathbb{R}^N_+$ denotes the claims. It is assumed that $0 \leq E \leq \sum_{i \in N} c_i$.

A *bankruptcy rule* is a function ψ satisfying that $\sum_{i \in N} \psi_i(E, c) = E$ and $0 \leq \psi_i(E, c) \leq c_i$ for each (E, c) and $i \in N$.

Some well-known bankruptcy rules are the following. The *proportional rule* divides the estate proportionally to the claims, i.e. $P(E, c) = \lambda c$, where λ satisfies $\sum_{i \in N} \lambda c_i = E$. The *constrained equal awards rule* makes awards as equal as possible, provided that no agent receives more than his claim, i.e. $CEA(E, c) = (\min\{\lambda, c_i\})_{i \in N}$, where $\sum_{i \in N} \min\{\lambda, c_i\} = E$.

A *multi-issue allocation (MIA) situation* is a 4-tuple (R, N, E, C), where $R = \{1, \ldots, r\}$ is the set of issues. $N = \{1, \ldots, n\}$ is the set of agents. $E \geq 0$ is the estate to be divided. $C = (c_{ki})_{k \in R, i \in N} \in \mathbb{R}^{R \times N}_+$ and c_{ki} represents the amount claimed by agent $i \in N$ on issue $k \in R$. We assume $0 \leq E \leq \sum_{k \in R} \sum_{i \in N} c_{ki}$.

Note that a bankruptcy situation is a *MIA* situation with $|R| = 1$.

A *multi-issue allocation (MIA) rule* f is a function that associates with each *MIA* situation (R, N, E, C) a matrix $f(R, N, E, C) \in \mathbb{R}^{R \times N}$ satisfying:

- $0 \leq f_{ki}(R,N,E,C) \leq c_{ki}$ for each $k \in R$ and each $i \in N$.
- $\sum_{k\in R}\sum_{i\in N} f_{ki}(R,N,E,C) = E$.

Note that we assume agents cannot make transfers among issues. If we consider the university example provided in the introduction, we would realize that the amount designated to a department for teaching (research) cannot be spent on research (teaching).

Lorenzo-Freire et al. (2009) define a two-stage procedure to obtain *MIA* rules from bankruptcy rules. They first apply a bankruptcy rule for dividing the estate among the issues. Later, the amount assigned to each issue is divided among the agents claiming on this issue.

Let ψ and ϕ be two bankruptcy rules and let (R,N,E,C) be a MIA situation. The *two-stage rule* $f^{\psi,\phi}(R,N,E,C)$ is the MIA rule obtained from the following two-stage procedure.

1. First stage. Consider the so-called bankruptcy situation among the issues (R,E,c^R), where $c^R = (c_1^R,\ldots,c_r^R) \in \mathbb{R}^R$ denotes the vector of total claims in the issues, i.e. $c_k^R = \sum_{i\in N} c_{ki}$ for each $k \in R$. The amount E is divided among the issues using the bankruptcy rule ψ.
2. Second stage. For each $k \in R$, consider the bankruptcy situation $(N,\psi_k(R,E,c^R),(c_{ki})_{i\in N})$ and apply the bankruptcy rule ϕ.

For each $k \in R$ and each $i \in N$,

$$f_{ki}^{\psi,\phi}(R,N,E,C) = \phi_i\left(N,\psi_k(R,E,c^R),(c_{ki})_{i\in N}\right).$$

3 The mixed rule

We define the two stage rule which divides among the issues following the CEA bankruptcy rule and inside each issue following the proportional bankruptcy rule. We give two axiomatic characterizations of $f^{CEA,P}$. The first one combine the results of Yeh (2006) and Chun (1988). The second one combine the results of Herrero and Villar (2002) and Chun (1988).

The *two-stage constrained equal awards - proportional rule* is the two-stage MIA rule $f^{\psi,\phi}$ where $\psi = CEA$ and $\phi = P$.

Then, for each (R,N,E,C), each $k \in R$, and each $i \in N$,

$$\begin{aligned} f_{ki}^{CEA,P}(R,N,E,C) &= P_i\left(N, CEA\left(R,E,c^R\right),(c_{ki})_{i\in N}\right) \\ &= \frac{c_{ki}}{c_k^R}\min\{\lambda, c_k^R\}, \end{aligned}$$

where λ satisfies $\sum_{k\in R}\min\{\lambda,c_k^R\} = E$.

We now extend bankruptcy properties to MIA properties. In some cases the definition is the same, as in composition down. In other cases, the properties are adapted by claiming it among the issues or inside each issue.

No advantageous transfer inside the issues $(NATI)$. Let (R,N,E,C), (R,N,E,C'), $k \in R$, and $M \subset N$, such that $\sum_{i\in M} c'_{ki} = \sum_{i\in M} c_{ki}$ and $c'_{li} = c_{li}$ when $l \in R\setminus\{k\}$ or $i \in N\setminus M$. Then for each $i \in N \setminus M$,

$$f_{ki}(R,N,E,C) = f_{ki}(R,N,E,C').$$

$NATI$ says that if a group of agents redistribute their claims inside an issue, the amount assigned to the other agents in this issue does not change.

Conditional full compensation among the issues (FCA). For each (R, N, E, C) and each $k \in R$ such that $\sum_{l \in R} \min \left\{ \sum_{i \in N} c_{li}, \sum_{i \in N} c_{ki} \right\} \leq E$,

$$\sum_{i \in N} f_{ki}(R, N, E, C) = \sum_{i \in N} c_{ki}.$$

Suppose that the estate is sufficient to satisfy all the claims in the issues truncated by the total claim in issue k. FCA says that the total amount allocated to issue k must coincide with its total claim.

Claims monotonicity among the issues (CMA). For each (R, N, E, C), each $k \in R$, and each (R, N, E, C') such that $\sum_{i \in N} c_{ki} \leq \sum_{i \in N} c'_{ki}$ and $c_{li} = c'_{li}$ for each $l \in R \setminus \{k\}$ and $i \in N$,

$$\sum_{i \in N} f_{ki}(R, N, E, C) \leq \sum_{i \in N} f_{ki}(R, N, E, C').$$

CMA says that if the total claim of an issue increases, then the total allocation to this issue can not decrease.

Composition down (CD). For each (R, N, E, C) and each $E' \in \mathbb{R}$ such that $E' \geq E$,

$$f(R, N, E, C) = f(R, N, E, f(R, N, E', C)).$$

Theorem.

(a) If $|N| \geq 3$, then $f^{CEA,P}$ is the unique rule satisfying FCA, CMA, and $NATI$.
(b) $f^{CEA,P}$ is the unique rule satisfying CD, FCA, and $NATI$.

Proof.

Using arguments similar to Bergantiños et al. (2009, 2010) it is possible to prove that $f^{CEA,P}$ satisfies the properties.

(a) We now prove that $f^{CEA,P}$ is the only one. Let f be a rule satisfying FCA, CMA, and $NATI$. The statement is a consequence of the following claims.

Claim 1. For each $k \in R$,

$$\sum_{i \in N} f_{ki} \left(R, N, E, c \right) = \sum_{i \in N} f_{ki}^{CEA,P} \left(R, N, E, c \right).$$

Claim 2. For each $k \in R$ and $i \in N$,

$$f_{ki}(R, N, E, (c_{lj})_{l \in R, j \in N}) = f_{ki}^{CEA,P}(R, N, E, (c_{lj})_{l \in R, j \in N}).$$

We avoid the proof of both claims.
(b) The proof of the uniqueness is similar to case (a). ■

Remark. The properties used in Theorem 1 are independent. We avoid the proof.

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References

Bergantiños, G., Lorenzo, L., Lorenzo-Freire, S., 2009. New characterizations of the constrained equal awards rule in multi-issue allocation situations. Mimeo, Universidade de Vigo. Available at webs.uvigo.es/gbergant/research.html.

Bergantiños, G., Lorenzo, L., Lorenzo-Freire, S., 2010. A characterization of the proportional rule in multi-issue allocation situations. Oper. Res. Lett. 38, 17-19.

Calleja, P., Borm, P., Hendrickx, R., 2005. Multi-issue allocation situations. Eur. J. Oper. Res. 164, 730-747.

Chun, Y., 1988. The proportional solution for rights problems. Math. Soc. Sci. 15, 231-246.

González-Alcón, C., Borm P., Hendrickx, R., 2007. A composite run to the bank rule for multi-issue allocation situations. Math. Method. Oper. Res. 65, 339-352.

Herrero, C., Villar, A., 2002. Sustainability in bankruptcy problems. TOP 10, 261-273.

Ju, B.G., Miyagawa, E., Sakai, T.,2007 . Non-manipulable division rules in claim problems and generalizations. J. Econ. Theory 132, 1-26.

Lorenzo-Freire, S., Casas-Méndez, B., Hendrickx, R., 2009. The two-stage constrained equal awards and losses rules for multi-issue allocation situations. TOP, doi 10.1007/s11750-009-0073-8.

Moreno-Ternero, J., 2009. The proportional rule for multi-issue bankruptcy problems. Econ. Bull. 29(1), 483-490.

Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Soc. Sci. 45, 249-297.

Yeh, C.H., 2006. Protective properties and the constrained equal awards for claims problems: a note. Soc. Choice Welfare 27, 221-230.

4 Appendix

Proof of Claim 1. Let $q = (q_{ki})_{k\in R, i\in N}$ be such that for each $k \in R$, $(q_{ki})_{i\in N}$ belongs to the simplex in $\mathbb{R}^N$. For each bankruptcy problem $(R, E, (x_k)_{k\in R})$, we define the bankruptcy rule f^q such that for each $k \in R$,

$$f_k^q(R, E, (x_k)_{k\in R}) = \sum_{i\in N} f_{ki}(R, N, E, (q_{ki}x_k)_{k\in R, i\in N}).$$

Since f is a MIA rule, f^q is a bankruptcy rule. We now prove that f^q satisfies FC and CM in bankruptcy situations.

- FC. Let $k \in R$ be such that $\sum\limits_{l\in R} \min\{x_l, x_k\} \leq E$. Thus, $\sum\limits_{l\in R} \min\left\{\sum\limits_{i\in N} q_{li}x_l, \sum\limits_{i\in N} q_{ki}x_k\right\}$ $\leq E$. Since f satisfies FCA, $f_{ki}(R, N, E, (q_{li}x_l)_{l\in R, i\in N}) = q_{ki}x_k$ for all $i \in N$. Hence, $f_k^q(R, E, (x_k)_{k\in R}) =$ $\sum\limits_{i\in N} q_{ki}x_k = x_k$.
- CM. Let $k, k' \in R$ be such that $x_k \geq x_{k'}$. Thus, $\sum\limits_{i\in N} q_{ki}x_k \geq \sum\limits_{i\in N} q_{k'i}x_{k'}$. Since f satisfies CMA,

$$\begin{aligned} f_k^q(R, E, (x_l)_{l\in R}) &= \sum_{i\in N} f_{ki}(R, N, E, (q_{li}x_l)_{l\in R, i\in N}) \\ &\geq \sum_{i\in N} f_{k'i}(R, N, E, (q_{li}x_l)_{l\in R, i\in N}) \\ &= f_{k'}^q(R, E, (x_l)_{l\in R}). \end{aligned}$$

Yeh (2006) proves that CEA is the unique bankruptcy rule satisfying FC and CM. Thus, $f^q = CEA$.

Consider $(x_k)_{k\in R}$ and q such that $x_k = \sum\limits_{i\in N} c_{ki}$ and $q_{ki} = \frac{c_{ki}}{x_k}$ for each $(k, i) \in R \times N$. Now, it is trivial to deduce that Claim 1 holds. ■

Proof of Claim 2. Let $d^k = (d_{lj})_{l\in R\setminus\{k\}, j\in N} \in \mathbb{R}_+^{R\setminus\{k\}\times N}$. For each bankruptcy problem $(N, E, (y_i)_{i\in N})$ we define the bankruptcy rule f^{d^k} such that for each $i \in N$

$$f_i^{d^k}(N, E, (y_j)_{j\in N}) = f_{ki}(R, N, E^d, (d_{lj})_{l\in R, j\in N})$$

where $d_{kj} = y_j$ and E^d satisfies $\min\left\{\lambda^d, \sum\limits_{i\in N} d_{ki}\right\} = E$ and $\sum\limits_{l\in R} \min\left\{\lambda^d, \sum\limits_{i\in N} d_{li}\right\} = E^d$. Then, $f^{d^k} = P$.

We first prove that f^{d^k} is well defined, namely, that it is a bankruptcy rule.

- Since f is a MIA rule, for each $i \in N$,

$$0 \leq f_{ki}(R, N, E^d, (d_{lj})_{l\in R, j\in N}) \leq d_{ki} = y_i.$$

- By Claim 1,

$$\begin{aligned} \sum_{i\in N} f_i^{d^k}(N, E, (y_j)_{j\in N}) &= \sum_{i\in N} f_{ki}(R, N, E^d, (d_{lj})_{l\in R, j\in N}) \\ &= \sum_{i\in N} f_{ki}^{CEA,P}(R, N, E^d, (d_{lj})_{l\in R, j\in N}) \\ &= \min\left\{\lambda, \sum_{i\in N} d_{ki}\right\} \end{aligned}$$

where $\sum_{l\in R}\min\left\{\lambda, \sum_{i\in N} d_{li}\right\} = E^d$. Thus, $\lambda = \lambda^d$ and hence, $\sum_{i\in N} f_i^{d^k}(N, E, (y_j)_{j\in N}) = \min\left\{\lambda^d, \sum_{i\in N} d_{ki}\right\} = E$.

We now prove that f^{d^k} satisfies NAT in bankruptcy situations. Let (N, E, y), (N, E, y'), and $M \subset N$ such that $y_i = y'_i$ when $i \in N \setminus M$ and $\sum_{i\in M} y'_i = \sum_{i\in M} y_i$. Thus, $d'_{li} = d_{li}$ when $l \in R \setminus \{k\}$ or $i \in N \setminus M$ and $\sum_{i\in M} d'_{ki} = \sum_{i\in M} y'_i = \sum_{i\in M} y_i = \sum_{i\in M} d_{ki}$.

For each $i \in N \setminus M$,

$$f_i^{d^k}(N, E, y') = f_{ki}(R, N, E^{d'}, (d'_{lj})_{l\in R, j\in N}).$$

Since f satisfies $NATI$,

$$f_{ki}(R, N, E^{d'}, (d'_{lj})_{l\in R, j\in N}) = f_{ki}(R, N, E^{d'}, (d_{lj})_{l\in R, j\in N}).$$

By Claim 1, it is easy to deduce that $E^d = E^{d'}$. Then,

$$f_{ki}(R, N, E^{d'}, (d_{lj})_{l\in R, j\in N}) = f_i^{d^k}(N, E, y).$$

Hence f^{d^k} satisfies NAT. Since P is the unique bankruptcy rule satisfying NAT (Chun, 1988), $f^{d^k} = P$.

We define d^k such that for each $l \in R\setminus\{k\}$ and each $j \in N$, $d_{lj} = c_{lj}$. Moreover, we take $y_j = d_{kj} = c_{kj}$ for all $j \in N$. Because of the definition of CEA it is easy to conclude that $CEA_k(R, E, c^R)^d = E$. Now

$$\begin{aligned} f_{ki}(R, N, E, c) &= f_i^{d^k}(N, CEA_k(R, E, c^R), (c_{kj})_{j\in N}) \\ &= P_i(N, CEA_k(R, E, c^R), (c_{kj})_{j\in N}) \\ &= f_{ki}^{CEA,P}(R, N, E, (c_{lj})_{l\in R, j\in N}). \blacksquare \end{aligned}$$

Proof of Remark. We prove that the properties used in Theorem 1 are independent. Let α be the bankruptcy rule in which we fulfill the claims by increasing order. Namely, given (E, c) such that $c_1 \leq c_2 \leq .... \leq c_n$ we can find $j \in N$ satisfying $\sum_{i\leq j} c_i \leq E < \sum_{i\leq j+1} c_i$. Thus, $\alpha_i(E, c) = c_i$ when $i \leq j$, $\alpha_{j+1}(E, c) = E - \sum_{i\leq j} c_i$ and $\alpha_i(E, c) = 0$ when $i > j + 1$.

Let δ be the bankruptcy rule defined by Herrero and Villar (2002).

Next tables show that the properties used in Theorem 1 are independent. The proofs are left to the reader.

Independence of the properties of Theorem 1 (a):

Properties / Rules	$f^{P,P}$	$f^{\alpha,P}$	$f^{CEA,CEA}$
FCA	No	Yes	Yes
CMA	Yes	No	Yes
$NATI$	Yes	Yes	No

Independence of the properties of Theorem 1 (b):

Properties / Rules	$f^{P,P}$	$f^{\delta,P}$	$f^{CEA,CEA}$
FCA	No	Yes	Yes
CD	Yes	No	Yes
$NATI$	Yes	Yes	No