Contracts for Uncertain Delivery[1]

João Correia-da-Silva[2]

FCT. Faculdade de Economia. Universidade do Porto. PORTUGAL.

Carlos Hervés-Beloso[3]

RGEA. Facultad de Económicas. Universidad de Vigo. SPAIN.

January 31th, 2005

Abstract. We propose the notion of objects of choice as uncertain consumption bundles, extending the formulation of Arrow (1953). Agents sign "*contracts for uncertain delivery*", which specify a list of alternative bundles, instead of a single one. This allows us to incorporate uncertainty and asymmetric information in the model of Arrow-Debreu. Relatively to the model of Radner (1968), efficiency of trade is increased and some "*no trade*" situations are avoided, while the classical results still hold: existence of core and competitive equilibrium, core convergence, welfare theorems, etc.

Keywords: Uncertainty, Asymmetric information, Private information, Contingent delivery, Radner equilibrium.

JEL Classification Numbers: C62, D51, D82, G10.

[1]We thank Mário Páscoa for an early insight and Jean Gabszewicz for comments and encouragement.

[2]João Correia-da-Silva (e-mail: joaocs@sapo.pt) acknowledges support by Research Grant SFRH/BD/11435/2002 from Fundação para a Ciência e Tecnologia (Portugal).

[3]Carlos Hervés-Beloso (e-mail: cherves@uvigo.es) acknowledges support by Research Grant BEC2003-09067-C04-01 from Ministerio de Ciencia y Tecnología (Spain) and FEDER.

1 Introduction

Uncertainty and information are crucial in virtually all economic decisions. They are a major source of the complexity that characterizes modern economies, and, in turn, of most criticisms of neoclassical economics and general equilibrium theory. Naturally, a large body of literature still under development seeks to incorporate uncertainty and asymmetric information in the model of Arrow and Debreu.

The theory of general equilibrium under uncertainty has developed upon the formulation of objects of choice as contingent consumption claims (Arrow, 1953). Under this formulation, besides being defined by their physical properties and their location in space and time, commodities can also be defined by their location in "state". For example, an "umbrella" in "rainy weather" and an "umbrella" in "sunny weather" are seen as two different commodities.

In the markets for contingent delivery, agents trade their state-dependent endowments for state-dependent consumption bundles. Radner (1968) extended the model of Arrow and Debreu to a setting of asymmetric information, by restricting agents to buy rights for delivery that can only be contingent upon events that they observe. As a consequence, each agent consumes the same in states of nature that she does not distinguish. That is, consumption is measurable with respect to the private information of the agents.

This restriction of measurability implies incentive compatibility. Whatever the state of nature that occurs, agents are always sure about the bundle that will be delivered to them, so they cannot be deceived. But incentive compatibility does not imply measurability, so this restriction may be seen as too severe.

In the modern economy, there are many situations in which the condition of measurability does not hold. Consider the contracts known as "options". The decision of the buyer to exercise or not the option may be based on the observation of an event that the seller cannot observe. So, the seller may end up with different bundles in states of nature that she would not be able to distinguish with her private information.

Having to consume the same in states of nature that the agent does not distinguish *a priori* seems a strong restriction. Relaxing this restriction could allow the achievement of better outcomes, in the sense of Pareto. But does it make sense for an agent to buy the right to receive different bundles in states of nature that she will not distinguish?

In undistinguished states of nature, the agent should have the same rights and obligations. This is a sensible idea and avoids problems of incentive compatibility. But to have the same rights and obligations does not imply consuming the same bundle. For example, an agent may buy the right to receive a "ham sandwich or cheese sandwich". The actual consumption is uncertain. Nevertheless, the agent values the right to consume this uncertain bundle since both possibilities are desirable.

We designate these multiple alternative bundles as "lists". And what we propose is that agents choose between contingent lists, instead of contingent bundles. If the contingency occurs, a contingent list gives an agent the right to receive one of the bundles in the list. Since the agent has no control over the selection of a bundle from the list for delivery, a list may be seen as an uncertain bundle and objects of choice as uncertain consumption bundles. This goes further than Arrow's (1953) formulation of objects of choice as contingent consumption bundles.

Underlying this formulation is the possibility of agents to sign "*contracts for uncertain delivery*". These contracts specify a list of bundles out of which a single one will be selected for delivery if the contingency occurs. Agents buy the right to receive one of the bundles in the list. For example, instead of buying the right to receive an “umbrella” if “weather is rainy” or the right to receive a “raincoat” if “weather is rainy”, agents may also buy the right to receive an “umbrella or raincoat” if “weather is rainy”. Observe that contracts for contingent delivery are also “*contracts for uncertain delivery*” with lists of a single element.

As agents are able to sign more general contracts, allowing contracts for uncertain delivery may be seen as opening additional markets. A supplier may not be able to guarantee the delivery of neither a “ham sandwich” nor a “cheese sandwich”, while being able to ensure the delivery of one of the two. In the Arrow-Radner framework this would lead to a “*no trade*” situation, while contracts for uncertain delivery allow trade to take place. This is what occurs in the example of the “generalized commodities” (section 3).

The problem is to assign prices and utilities to uncertain bundles. Consider a contract for the delivery of “ham sandwich or cheese sandwich”. The right to receive a “ham sandwich or cheese sandwich” is weaker than the right to receive a “ham sandwich”. Observe that the first does not imply the second, while delivery of a “ham sandwich” implies delivery of a “ham sandwich or cheese sandwich”. Thus, uncertain delivery of “ham sandwich or cheese sandwich” should not be more expensive nor give more utility than the delivery of a “ham sandwich”. If it were more expensive, there would be an opportunity for arbitrage. An intermediary could buy a “ham sandwich” and sell it as a “ham sandwich or cheese sandwich” with profit. If it gave more utility, a simple “ham sandwich” would not be sold. All sellers would prefer to sell a “ham sandwich or cheese sandwich” and always deliver a “ham sandwich”.

For analytical convenience, in our model agents do not buy lists (uncertain bundles). They buy contingent bundles, as in the model of Radner, but these do not need to be measurable with respect to the information of the agent. What we do is an extension of the domain in which preferences are defined to include also the bundles that are not measurable. We argue that nothing essential is lost relatively to a model where agents buy lists (see section 6). For an agent, a list is equivalent to a bundle that is not measurable with respect to her information. This equivalence is illustrated in the example that follows.

Consider three possible states of nature: $\Omega = \{\omega_1, \omega_2, \omega_3\}$. An agent does not distinguish ω_1 from ω_2, but may select a random consumption bundle that delivers x_1 in ω_1, x_2 in ω_2 and x_3 in ω_3. With $x_1 \neq x_2$, this consumption bundle is not measurable with respect to her information. In ω_1 and ω_2, the agent will have to accept delivery of x_1 or x_2. She may prefer x_1 and the real state of nature may be ω_1, but since she cannot prove that the state of nature is ω_1, she has to accept x_2 if it is the bundle that is delivered. In ω_1 and ω_2, she receives an uncertain bundle that we denote as $(x_1 \vee x_2)$. Instead of writing the consumption bundle as $x = (x_1, x_2, x_3)$, from the perspective of the agent it would be more adequate to use the notation $\bar{x} = [(x_1 \vee x_2), (x_1 \vee x_2), x_3]$. Observe that this construction implies measurability of the vector of contingent lists with respect to the information of the agent.

In this setting, assigning prices to the lists becomes straightforward. The information of the agent is $P = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. She can obtain the list $\bar{x} = [(x_1 \vee x_2), (x_1 \vee x_2), x_3]$ by buying any of the following bundles: $x_a = (x_1, x_1, x_3)$, $x_b = (x_1, x_2, x_3)$, $x_c = (x_2, x_1, x_3)$, or $x_d = (x_2, x_2, x_3)$. It only makes sense to buy the cheapest of these bundles, so the price of a list is actually the price of the cheapest bundle that is equivalent, for the agent, to the list.

Another issue concerns the preferences of an agent regarding a set of bundles out of which she will obtain her consumption bundle. What is the utility of $(x_1 \vee x_2)$? Our proposal is that the utility of an uncertain bundle (a list) is equal to the utility of the worst possible outcome.

The following three assumptions suffice to arrive at these "pessimistic preferences": (1) agents are neutral with respect to uncertainty, that is, they are indifferent between a bundle and a list in which all the alternatives have the same utility as that bundle; (2) the substitution of a bundle in a list for another with greater utility does not decrease the utility of the list; and (3) agents are indifferent between a bundle x_1 and a list with x_1 and x_T, where x_T is greater than the total resources in the economy.[4]

If consumption is measurable with respect to the information of the agents, the lists that correspond to the events that the agent observes (sets of the partition of information) have only one element. To see this, let the information of an agent be $P = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, and consider the measurable consumption bundle $x = (x_1, x_1, x_3)$. The correspondent list is $\bar{x} = [(x_1 \vee x_1), (x_1 \vee x_1), x_3] = (x_1, x_1, x_3)$.

With lists having only one element, the pessimistic expected utility of consumption bundles that are measurable is equal to the classical expected utility. What is done is an enlargement of the domain in which utility is defined to include also the non-measurable bundles, but the values in the space of measurable bundles are preserved.

Assuming that the seller knows the preferences of the buyer, we should expect the bundles in the list to have equal utility for the buyer. Suppose that an agent buys the uncertain bundle $(x_1 \vee x_2)$ and $u(x_2) > u(x_1)$. The utility of this bundle is equal to the utility of the worst alternative: $u(x_1 \vee x_2) = u(x_1)$. So, the seller could substitute x_2 by a fraction $x_2' = (1 - \delta)x_2$ such that $u(x_2') = u(x_1)$. The utility of this bundle, $(x_1 \vee x_2')$ would still be equal to $u(x_1)$, and the seller would retain a valuable bundle, δx_2. In sum, it does not make sense for the seller to offer alternatives with different utility, because the buyer focuses only on the worst alternative.

The assumption that the market knows the preferences of the agents is common in general equilibrium, as the Walrasian auctioneer uses the preferences of the

[4]They will always receive x_1, and never x_T, so the indifference between x_1 and $(x_1 \vee x_T)$ is actually realistic.

agents to arrive at the equilibrium prices. In fact, we arrive naturally at these equal utility alternatives in equilibrium. In our model, a property of competitive equilibrium is that the consumption bundles have measurable utility. That is, the lists selected by the agents for each event that they observe have bundles with equal utility.

While the intuition for the concept of rational expectations equilibrium was the idea that "*agents cannot be fooled*", our equilibrium concept can be justified with an opposite idea: "*the market cannot be fooled*". It has also some relation to Murphy's law: "*if anything can go wrong, it will*".

With alternatives having the same utility, after observing the state of nature, the buyer is sure about the utility that she is entitled to receive. There is uncertainty about the consumption bundle, but not about the utility that is obtained. Agents cannot be deceived to receive consumption bundles with lower utility. So, under the hypothesis of neutrality towards uncertainty, agents do not care about having been deceived or not.

For example, consider two agents and two states of nature where the difference between ω_1 and ω_2 is the result of a toss of a coin by agent B, an event that agent A cannot observe. Agent A may accept to pay "\$1" to receive a "ham sandwich" from agent B if the result is heads, and a "cheese sandwich" if it is tails. Being indifferent between the two sandwiches, the impossibility of distinguishing the two states *a priori* is not a problem. Agent A does not fear being "tricked", as receiving a sandwich is guaranteed.

Since agents cannot be deceived, problems of incentive compatibility do not arise, independently of the beliefs of the buyer relatively to the preferences of the seller. In sum, the consideration of this type of contracts allows us to relax in a natural way the measurability assumption, while preserving incentive compatibility. This enlarges the space of allocations, improving the efficiency of exchange, relatively to the Walrasian expectations equilibrium (Radner) solutions.

The inclusion of these contracts for uncertain delivery in the general equilibrium theory is based on the pessimistic preferences just described. With these preferences, we incorporate uncertainty and asymmetric information in the model of Arrow-Debreu in a direction that seems more satisfactory than the proposal of Radner. This is accomplished by an extension of the utility functions from the domain of measurable consumption bundles to the whole space. The essential properties of the primitive utility functions are preserved. As a consequence, all the results in the literature still hold: existence of competitive equilibrium, existence of core, core convergence, continuity properties, welfare theorems, etc.

The paper is organized as follows: in section 2 contracts for uncertain delivery are introduced; sections 3 and 4 consist of the examples that motivate the paper; in section 5 we derive the preferences over uncertain bundles; in section 6 the model of general equilibrium with asymmetric information is presented; and, finally, in section 7 we conclude the paper with several results.

2 Contracts for Uncertain Delivery

A "*contract for contingent delivery*" (Arrow, 1953) specifies a bundle to be delivered if a certain event occurs. We extend this notion to allow for the delivery of an uncertain bundle. Instead of a single bundle, a list of possible bundles is specified. "*Contracts for uncertain delivery*" stipulate that the seller must deliver to the buyer one of the bundles in a list.

The buyer is certain about receiving one of the bundles in the list, but has no control over the selection process. She does not know the probabilities of receiving each of the bundles (if the probabilities were known, a better designation would be of a "*contract for risky delivery*"). We designate as "seller" the agent that selects the bundle to be delivered, and as "buyer" the agent that obtains the right to receive one of the bundles in a list.

A generalized contract for contingent delivery specifies: the price of the contract, a list of possible bundles to be delivered, and the states of nature in which delivery takes place. Contracts commonly known as "options" are included in this definition.

These contracts allow an increase in the efficiency of trade, relatively to the regular contingent contracts. In the next two sections, we present different situations in which trade is not possible with contingent contracts. As a consequence, Walrasian expectations equilibrium is a "*no trade*" situation. But contracts for uncertain delivery allow trade to take place, allowing welfare improvements in the sense of Pareto.

3 Example: Generalized Commodities

In the example that follows, contracts for uncertain delivery allow agents to reach the full information outcome, while if they were restricted to contingent contracts, "*no trade*" would be an equilibrium.

This economy has two agents and four commodities: "ham sandwiches", "cheese sandwiches", "orange juices" and "apple juices".

There are four states of nature, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

- In ω_1, agent A is endowed with two "ham sandwiches" and agent B with two "orange juices": $e_A(\omega_1) = (2, 0, 0, 0)$ and $e_B(\omega_1) = (0, 0, 2, 0)$;
- In ω_2, agent A is endowed with two "ham sandwiches" and agent B with two "apple juices": $e_A(\omega_2) = (2, 0, 0, 0)$ and $e_B(\omega_2) = (0, 0, 0, 2)$;
- In ω_3, agent A is endowed with two "cheese sandwiches" and agent B with two "orange juices": $e_A(\omega_3) = (0, 2, 0, 0)$ and $e_B(\omega_3) = (0, 0, 2, 0)$;
- In ω_4, agent A is endowed with two "cheese sandwiches" and agent B with two "apple juices": $e_A(\omega_4) = (0, 2, 0, 0)$ and $e_B(\omega_4) = (0, 0, 0, 2)$.

Each agent observes only its endowments. Their information partitions are:

$P_A = \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}$ and $P_B = \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}$.

Their preferences are the same in every state. The sandwiches are perfect substitutes, as well as the juices. But agents like to mix sandwiches and juices. A Cobb-Douglas utility function describes their tastes:

$U = (s_h + s_c)^{0.5} \cdot (j_o + j_a)^{0.5}$.

Observe that agent A is not be able to guarantee the delivery of neither a "ham sandwich" nor a "cheese sandwich". Agent B has the same problem with guaranteeing delivery of "orange juice" or "apple juice". As a consequence, we have "*no trade*"! To see this, suppose that agent A consumed some quantity of "orange juice" in ω_1. She would have to consume the same in ω_2 (because she does not distinguish ω_1 and ω_2), but in ω_2 there isn't any "orange juice". This is a contradiction.[5]

The "*no trade*" situation may be overcome if we consider that an agent can buy a sandwich (or a juice). We model this with contracts for the delivery of one out of two bundles. One of the agents sells a "ham sandwich or cheese sandwich", while the other sells an "orange juice or apple juice". Since agent A is able to ensure the delivery of a "sandwich" and agent B is able to ensure the delivery of a "juice", contracts for uncertain delivery allow them to attain the optimal outcome. Agents trade a sandwich for a drink, thus, in every state of nature each agent consumes a sandwich and a drink. The symmetric optimal outcome is, therefore:

$$x_A = \begin{cases} (1,0,1,0) \text{ in } \omega_1, \\ (1,0,0,1) \text{ in } \omega_2, \\ (0,1,1,0) \text{ in } \omega_3, \\ (0,1,0,1) \text{ in } \omega_4. \end{cases} \qquad x_B = \begin{cases} (1,0,1,0) \text{ in } \omega_1, \\ (1,0,0,1) \text{ in } \omega_2, \\ (0,1,1,0) \text{ in } \omega_3, \\ (0,1,0,1) \text{ in } \omega_4. \end{cases}$$

Both agents obtain an utility that is equal to 1 in all states of nature, so expected pessimistic utility is equal to 1. This constitutes an improvement in the sense of Pareto relatively to the Walrasian expectations equilibrium solution, which resulted in an utility of zero to both agents.

Together with the price vector $p = \frac{1}{24}[(1,2,1,2);(1,2,2,1);(2,1,1,2);(2,1,2,1)]$, this allocation is a competitive equilibrium of the economy with uncertain

[5] We can assume strictly positive endowments, substituting every zero for a small ϵ, and reach the same conclusions.

delivery.

Allocations in the core are of the form:

$$x_A = \begin{cases} (a, 0, 2-b, 0) \text{ in } \omega_1, \\ (a, 0, 0, 2-b) \text{ in } \omega_1, \\ (0, a, 2-b, 0) \text{ in } \omega_1, \\ (0, a, 0, 2-b) \text{ in } \omega_4. \end{cases} \qquad x_B = \begin{cases} (2-a, 0, b, 0) \text{ in } \omega_1, \\ (2-a, 0, 0, b) \text{ in } \omega_1, \\ (0, 2-a, b, 0) \text{ in } \omega_1, \\ (0, 2-a, 0, b) \text{ in } \omega_4. \end{cases}$$

In states of nature that an agent does not distinguish "a priori" the consumption vectors are different, but note that the correspondent utility is always the same.

4 Example: Risk Sharing

Consider two agents with information fields that do not allow them to make contingent contracts. The space of states of nature is $\{\omega_1, \omega_2, \omega_3\}$, and agents have the information partitions: $P_A = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ and $P_B = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$.

Suppose that both agents have the following prior probabilities: 0.499 for ω_1 and ω_3 and 0.002 for ω_2.

There are two commodities, and the initial endowments vary with the state of nature that occurs:

$$e_A = \begin{cases} (199, 100) \text{ in } \omega_1, \\ (1, 100) \text{ in } \{\omega_2, \omega_3\}. \end{cases} \qquad e_B = \begin{cases} (1, 100) \text{ in } \{\omega_1, \omega_2\}, \\ (199, 100) \text{ in } \omega_3. \end{cases}$$

The agents have equal preferences, constant across states of nature. Good 1 has a diminishing marginal utility and good 2 (may be interpreted as money) has constant marginal utility:

$$U_A(x_1, x_2) = U_B(x_1, x_2) = 10\sqrt{x_1} + x_2.$$

Observe that the game is symmetric. Agent A wants to sell good 1 in ω_1 and to buy in $\{\omega_2, \omega_3\}$. Agent B wants good 1 in $\{\omega_1, \omega_2\}$ and to sell it in ω_3.

The total resources in the economy are:

$$e_{total} = \begin{cases} (200, 200) \text{ in } \omega_1, \\ (2, 200) \text{ in } \omega_2, \\ (200, 200) \text{ in } \omega_3. \end{cases}$$

In the least probable state, ω_2, physical feasibility implies that $x_1^A + x_1^B = 2$. This restriction is crucial. Now let's analyze first the symmetric possibilities.

(1 - with symmetry) If $x_1^A(\omega_2) = x_1^B(\omega_2) = 1$, measurability implies that $x_1^A = 1$ in ω_3 and $x_1^B = 1$ in ω_1. Agents retain their endowments, so there is no trade. The resulting utilities are:

$$U_1 = U_2 = 0.499 \cdot (10\sqrt{199} + 100) + 0.501 \cdot 110 = 0.499 \cdot 241 + 0.501 \cdot 110 = 175.$$

(2 - without symmetry) Without loss of generality:

$$\begin{cases} x_1^A(\omega_2) = x_1^A(\omega_3) = 1 + e, \\ x_1^B(\omega_2) = x_1^B(\omega_1) = 1 - e. \end{cases}$$

Physical feasibility implies that:

$$\begin{cases} x_1^A(\omega_1) \leq 200 - x_1^B(\omega_1) \leq 199 + e, \\ x_1^B(\omega_3) \leq 200 - x_1^A(\omega_3) \leq 199 - e. \end{cases}$$

It may be seen that the only measurable and efficient allocations are of the form:

$$\begin{cases} x^A(\omega_1) = (199 + e, 100 - p), \\ x^A(\omega_2) = (1 + e, 100 - p), \\ x^A(\omega_3) = (1 + e, 100 - p). \end{cases} \qquad \begin{cases} x^B(\omega_1) = (1 - e, 100 + p), \\ x^B(\omega_2) = (1 - e, 100 + p), \\ x^B(\omega_3) = (199 - e, 100 + p). \end{cases}$$

Trade is constant across states of nature. To receive an additional quantity, e, of good 1, agent A pays p units of good 2. Since the utility of good 2 is linear (agents can transfer utility through good 2), this assumption can be made without loss of generality. Then:

$$U_A = 0.499 \cdot (10 \cdot \sqrt{199 + e} + 100 - p) + 0.501 \cdot (10 \cdot \sqrt{1 + e} + 100 - p) =$$

$$= 4.99 \cdot \sqrt{199 + e} + 5.01 \cdot \sqrt{1 + e} + 100 - p.$$

$$U_B = 0.499 \cdot (10 \cdot \sqrt{199 - e}) + 0.501 \cdot (10 \cdot \sqrt{1 - e}) + 100 + p =$$

$$= 4.99 \cdot \sqrt{199 - e} + 5.01 \cdot \sqrt{1 - e} + 100 + p.$$

$$U_A + U_B = 4.99 \cdot (\sqrt{199 + e} + \sqrt{199 - e}) + 5.01 \cdot (\sqrt{1 + e} + \sqrt{1 - e}) + 200.$$

$$\frac{d(U_A + U_B)}{de} = 4.99 \cdot [\frac{1/2}{\sqrt{199 + e}} - \frac{1/2}{\sqrt{199 - e}}] + 5.01 \cdot [\frac{1/2}{\sqrt{1 + e}} - \frac{1/2}{\sqrt{1 - e}}] < 0$$

It is not possible to increase the sum of the utilities, therefore Pareto improvements are not possible. The asymmetric solution is even worst. We have "*no trade*" in this economy. The only "core" allocation corresponds to the initial endowments.

Can the agents improve this situation? In the symmetric allocations, each agent gets $(1, 100)$ in ω_2. The correspondent utilities are $u_A = u_B = 110$. An allocation with measurable utility for agent A must have the same utility in ω_3:

$$10\sqrt{x_1^A(\omega_3)} + x_2^A(\omega_3) = 110 \Rightarrow x_2^A(\omega_3) = 110 - 10\sqrt{x_1^A(\omega_3)}.$$

Thus, $x^A(\omega_3)$ must be of the form:

$$x^A(\omega_3) = (X, 110 - 10\sqrt{X}).$$

Without waste of resources, we have $x^B(\omega_3) = (200 - X, 90 + 10\sqrt{X})$. By symmetry, $x^B(\omega_1) = (X, 110 - 10\sqrt{X})$ and $x^A(\omega_1) = (200 - X, 90 + 10\sqrt{X})$.

The utility of agent A in $\{\omega_2, \omega_3\}$ is given and equal to 110. To arrive at an optimal solution, it is enough to maximize utility in ω_1:

$$U = 10\sqrt{200 - X} + 90 + 10\sqrt{X} \Rightarrow$$

$$\Rightarrow U' = -5 \cdot (200 - X)^{-1/2} + 5 \cdot X^{-1/2}.$$

$$U' = 0 \Rightarrow (200 - X)^{-1/2} = X^{-1/2} \Rightarrow$$

$$\Rightarrow X = 100.$$

$$u_A(\omega_1) = 10 \cdot \sqrt{100} + 90 + 10\sqrt{100} = 290.$$

$$U_A = U_B = 0.499 \cdot 290 + 0.501 \cdot 110 = 200.$$

The symmetric optimal solution is:

$$x^A = \begin{cases} (100, 190) \text{ in } \omega_1, \\ (1, 100) \text{ in } \omega_2, \\ (100, 10), \omega_3. \end{cases} \qquad u^A = \begin{cases} 290 \text{ in } \omega_1, \\ 110 \text{ in } \{\omega_2, \omega_3\}. \end{cases}$$

$$x^B = \begin{cases} (100, 10) \text{ in } \omega_1, \\ (1, 100) \text{ in } \omega_2, \\ (100, 190), \omega_3. \end{cases} \qquad u^B = \begin{cases} 110 \text{ in } \{\omega_1, \omega_2\}, \\ 290 \text{ in } \omega_3. \end{cases}$$

They can obtain this allocation by signing a contract under which, in every state of nature, each agent would deliver to the other one of two bundles: $(99, -90)$ or $(0, 0)$. It is straightforward to see that agents would deliver $(99, -90)$ if their endowments are $(199, 100)$, ending up with (100,190) in that state of nature.

This solution can also be achieved as a competitive equilibrium with the prevailing price vector $p = [(1, 2); \frac{2}{499}(10, 2); (1, 2)]$, leading agents to select the non-measurable bundles x^A and x^B.

The resulting expected utility is close to 200, higher than the 175 which correspond to the classical solution. Again, the possibility of signing these contracts allows a Pareto improvement in the exchange economy.

5 Preferences over uncertain bundles

A crucial matter concerns the preferences of agents regarding uncertain bundles (lists). What is the utility of $(x_1 \vee x_2)$? Equivalently, suppose that an agent signs a contract that may give him different consumption bundles in states of nature that she does not distinguish. If these bundles have different utilities, then what utility should she assign to the contract?

Start by assuming that agents are neutral with respect to uncertainty, in the sense made precise below:

Assumption 1 (Uncertainty neutrality)

$$\forall x_1, ..., x_k : u(x_1) = ... = u(x_k) \Rightarrow u(x_1 \vee ... \vee x_k) = u(x_1) = ... = u(x_k).$$

Assume also a kind of monotonicity.

Assumption 2 (Weak monotonicity)

$$u(x_j) \geq u(y_j) \ , \ \forall j = 1, ..., k \ \Rightarrow \ u(x_1 \vee ... \vee x_k) \geq u(y_1 \vee ... \vee y_k).$$

These two assumptions imply that the utility of a list is greater than the utility of the worst possibility. To see this, assume, w.l.o.g., that the least preferred bundle in a list $\bar{x} = (x_1 \vee ... \vee x_k)$ is x_1. Continuous preferences imply that if $u(x_j) > u(x_1)$, then there exists $\delta_j > 0$ such that $x'_j = (1 - \delta_j)x_j$ and $u(x'_j) = u(x_1)$. That is, if the outcomes have different utilities, then we can remove a positive fraction from some of them to arrive at a modified list $\bar{x}'$ in which all the alternatives have the same utility as x_1. By monotonicity (A.2), $u(\bar{x}) \geq u(\bar{x}')$. By uncertainty neutrality (A.1), the utility of the list $\bar{x}'$ is equal to the utility of x_1. As a consequence, $u(\bar{x}) \geq u(x_1)$. We arrive at Proposition 1, which states that the utility of a list cannot be lower than the utility of the worst outcome.

Proposition 1 (Rationality)

$$\forall x_1, ..., x_k : u(x_1 \vee ... \vee x_k) \geq \min_{j=1,...,k} u(x_j).$$

Observe that receiving x_1 guarantees that the right to receive $(x_1 \vee x_2)$ is satisfied. This advises the agent not to assign a higher utility to the uncertain bundle. Suppose that uncertainty is between three possible bundles: a "ham sandwich", a "cheese sandwich" and "$1 million". Since it is the (hypothetic) seller that selects the bundle after the observation of the state of nature, the buyer should simply ignore the possibility of receiving "$1 million".

We need only a weaker assumption: an agent is indifferent between a bundle x_1 and a list with x_1 and x_T, where x_T is greater than the total resources in the economy. The agent realistically expects to receive always x_1, and never x_T, so she is indifferent between x_1 and $(x_1 \vee x_T)$.

Assumption 3 (Realism) *Let* $x_T > \sum e$.

$$\forall x_1, ..., x_k : u(x_1 \vee ... \vee x_k \vee x_T) = u(x_1 \vee ... \vee x_k).$$

If introducing x_T does not increase the utility of the list, then, by our monotonicity assumption (A.2), we know that introducing an alternative that is less attractive than x_T also does not. So, the inclusion of additional alternative bundles (which may never be selected) does not increase the utility of the list.

Proposition 2 (Irrelevance of better alternatives)

$$\forall x_1, ..., x_k, x_{k+1} : u(x_1 \vee ... \vee x_k \vee x_{k+1}) \leq u(x_1 \vee ... \vee x_k).$$

Without loss of generality, suppose that x_1 is the worst outcome in the list. From A.2 and A.3, we have: $u(x_1 \vee x_2 \vee ... \vee x_k) \leq u(x_1 \vee x_T \vee ... \vee x_T) = u(x_1)$. Rationality (P.1) implies that we arrive at "pessimistic preferences". Agents are indifferent between the uncertain bundle and the worst possibility.

Proposition 3 (Pessimistic preferences)

$$\forall x_1, ..., x_k : u(x_1 \vee ... \vee x_k) = \min_{j=1,...,k} u(x_j).$$

In sum, "uncertainty neutrality" (A.1), "weak monotonicity" (A.2) and "realism" (A.3) imply what we designate as "pessimistic preferences" (P.3).

Observe that the pessimistic expected utility of consumption bundles that are measurable with respect to the information of the agents is equal to the classical expected utility. What is done is an enlargement of the domain in which utility is defined to include also the non-measurable bundles, preserving the values in the space of measurable bundles.

6 General Equilibrium with Uncertainty

Our model of an exchange economy with differential information assumes a finite number of agents, commodities and states of nature. The economy extends over two time periods. In the first, agents trade their endowments for multiple alternative deliveries (lists) that may be contingent on the state of nature that occurs in the second period (*ex-ante* contract arrangement). In the second period they receive and consume one of the alternative bundles correspondent to the events that they observe.

This suggests that each agent selects a list, contingent on events that she can observe. Such object of choice can be denoted as $\bar{x} \in \mathbb{R}^{\Omega K l}$ or $\bar{x} : \Omega \longrightarrow \mathbb{R}^{Kl}$, assuming that lists have a maximum of K alternatives.[6]

In our model, agent don't select lists. What they select are contingent bundles which need not be measurable with respect to their information. These bundles are, in turn, equivalent to lists. For example, suppose that $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and that the agent's partition of information is $P_i = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. The agent may select a consumption bundle that is not P_i-measurable, $x = (x_1, x_2, x_3)$. If the agent observes $\{\omega_1, \omega_2\}$, she has the right to receive x_1 or x_2, while the observation of ω_3 ensures consumption of x_3. So, from the perspective of the agent, this bundle is seen as the following P_i-measurable uncertain bundle: $x = [(x_1 \vee x_2), (x_1 \vee x_2), x_3]$.

This transformation from choice between lists to choice between bundles may seem restrictive, but nothing essential is lost. Suppose that lists are restricted to a maximum of K alternatives. Replicate the economy with Ω states of nature to transform each state into K identical states. This economy is equivalent to the

[6]To write a list with only two alternatives, complete it with repeated entries: $\bar{x} = (x_1 \vee x_2 \vee x_2 \vee ... \vee x_2)$.

original economy. But for each original state of nature there are K identical states. So, agents can select any consumption list with a maximum of K alternatives for each original state of nature, by selecting different consumption bundles in the correspondent K states of nature of the replicated economy.

With prices being equal in the K subdivisions of each state of nature, agents have no incentive to select different bundles in subdivisions of the same state of nature. They are fully satisfied with a single bundle for each state of nature. Therefore, selecting a bundle instead of a list is not restrictive at all. The consumption of an agent is written as a vector that is not necessarily P_i-measurable: $x \in \mathbb{R}_+^{\Omega l}$ or $x : \Omega \longrightarrow \mathbb{R}_+^l$.

In the differential information economy, $\mathcal{E} \equiv (e_i, u_i, P_i, q_i)_{i=1}^n$, for each agent i:

- A partition of Ω, P_i, generates its private information. Sets that belong to P_i are denoted A_i. We also denote the set of states of nature that agent i does not distinguish from ω_j by $P_i(\omega_j)$.
- Agents assign subjective probabilities to the different events that they observe. To each set $A_i^k \in P_i$ corresponds a probability $q_i(A_i^k)$, with $\sum_k q_i(A_i^k) = 1$.
- $u_i : \Omega \times \mathbb{R}_+^l \to \mathbb{R}_+$ is the random utility function. For all ω_j, the function $u_i^{\omega_j} = u_i(\omega_j, \cdot) : \mathbb{R}_+^l \to \mathbb{R}_+$ is continuous, weakly monotone and concave. For all $x_i \in \mathbb{R}_+^l, u_i(\cdot, x) : \Omega \to \mathbb{R}_+$ is P_i-measurable.[7]
- $e_i \in \mathbb{R}_+^{\Omega l}$, represents the random initial endowments. It is P_i-measurable and strictly positive: $e_i(\omega) \gg 0$ for all $\omega \in \Omega$.[8]

[7]It is equivalent to consider $u_i : P_i \times \mathbb{R}_+^l \to \mathbb{R}_+$ such that: $\forall A_i^k, u_i^{A_i^k} = u_i(A_i^k, \cdot) : \mathbb{R}_+^l \to \mathbb{R}_+$ is continuous, weakly monotone and concave.

[8]This can be replaced by $\sum_{i=1}^n e_i(\omega) \gg 0$ for all $\omega \in \Omega$, together with "*irreducibility*" (i.e., the endowment of every coalition is desired).

The *ex-ante* consumption bundle of agent i, denoted by $x_i \in \mathbb{R}_+^{\Omega l}$, is not necessarily P_i-measurable. From the perspective of the agent, consumption may be uncertain. Evaluation of these uncertain bundles in terms of utility is based on the analysis developed in section 5.

Agents seek to maximize their expected utility. They pessimistically evaluate their *interim* utility in each set of their information partition as:

$$v_i(x_i, A_i^k) = \min_{\omega \in A_i^k} u_i\left(x_i(\omega), \omega\right).$$

The expected utility is the expected *interim* utility:

$$U_i(x_i) = \sum_{A_i \in P_i} q_i(A_i)\ v_i(x_i, A_i).$$

From the properties of the random utility functions, u_i, it is shown below that the expected utility function is also concave.

$$\begin{aligned}
&U_i(\lambda x_i + (1-\lambda)y_i) = \sum_{A_i \in P_i} q_i(A_i)\ v_i(\lambda x_i + (1-\lambda)y_i, A_i) = \\
&= \sum_{A_i \in P_i} q_i(A_i)\ \min_{\omega \in A_i} [u_i(\lambda x_i(\omega) + (1-\lambda)y_i(\omega), \omega)] \geq \\
&\geq \sum_{A_i \in P_i} q_i(A_i)\ \min_{\omega \in A_i} [\lambda u_i(x_i(\omega), \omega) + (1-\lambda)u_i(y_i(\omega), \omega)] \geq \\
&\geq \sum_{A_i \in P_i} q_i(A_i)\ \min_{\omega \in A_i} [\lambda u_i(x_i(\omega), \omega)] + \sum_{A_i \in P_i} q_i(A_i)\ \min_{\omega \in A_i} [(1-\lambda)u_i(y_i(\omega), \omega)] = \\
&= \lambda \sum_{A_i \in P_i} q_i(A_i)\ v_i(x_i, A_i) + (1-\lambda) \sum_{A_i \in P_i} q_i(A_i)\ v_i(y_i, A_i) = \\
&= \lambda U_i(x_i) + (1-\lambda)U_i(y_i).
\end{aligned}$$

Note that if x is P_i-measurable, then $\bar{x} = x$ and $\bar{U}(\bar{x}) = U(x)$. Of course that this occurs when agents are perfectly informed. With symmetric information, the transformed model that we present below is no different from the classical model of Arrow and Debreu.

The differential information economy is transformed in an Arrow-Debreu economy, $\mathcal{E}_{AD} \equiv (e_i, U_i)_{i=1}^n$, where, for each agent i:

- The utility function, $U_i : \mathbb{R}_+^{\Omega l} \rightarrow \mathbb{R}_+$, is continuous, weakly monotone and concave.

- The vector of initial endowments, $e_i \in \mathbb{R}_+^{\Omega l}$, is strictly positive.

Everything is as in the model of Arrow and Debreu. With "free disposal", the condition of physical feasibility is:

$$\sum x \leq \sum e \Leftrightarrow \forall \omega : \sum x(\omega) \leq \sum e(\omega).$$

A "price system" is a non-zero function $p : \Omega \rightarrow \mathbb{R}_+^{\ell}$. We restrict the price functions to the simplex of $\mathbb{R}^{\Omega l}$, that is:

$$\sum_{\omega \in \Omega} \sum_{j=1,\dots,l} p_j(\Omega) = 1.$$

The "budget set" of agent i is given by:

$$B_i(p, e_i) = \left\{ x_i \in \mathbb{R}^{\Omega l}, \text{ such that } \sum_{\Omega} p(\omega) x_i(\omega) \leq \sum_{\Omega} p(\omega) e_i(\omega) \right\}.$$

A pair (p, x) is a competitive equilibrium of the economy with uncertain delivery if p is a price system and $x = (x_1, ..., x_n) \in \mathbb{R}^{n\Omega l}$ is a feasible allocation such that, for every i, x_i maximizes U_i on $B_i(p, e_i)$.

Uncertainty and asymmetric information are introduced in the model of Arrow and Debreu by a simple transformation of the preferences. This transformation preserves the properties of weak monotonicity and concaveness. Everything else in the model remains unchanged. We have, therefore, existence of equilibrium guaranteed. In fact, virtually all the results in the literature still hold: existence of core and competitive equilibrium, core convergence, welfare theorems, etc.

7 Some Results

In our model, a property of competitive equilibrium allocations is that in states of nature that an agent does not distinguish, the utility of the contingent bundles is the same. This means that instead of the widely used restriction of measurable consumption, we have a restriction of measurable utility arising naturally.

Theorem 1 *Let x be a competitive equilibrium allocation.*

$$\omega' \in P_i(\omega) \Rightarrow u_i^{\omega}(x_i(\omega)) = u_i^{\omega'}(x_i(\omega')).$$

Proof. Recall that for any $\omega' \in P_i(\omega)$, we have $u_i^{\omega} = u_i^{\omega'}$. Now suppose that for some $\omega' \in P_i(\omega)$, we have different utilities, that is: $u_i^{\omega}(x_i(\omega)) > u_i^{\omega'}(x_i(\omega'))$. Then, there exists some positive δ such that $u_i^{\omega}(\delta \cdot x_i(\omega)) = u_i^{\omega'}(x_i(\omega'))$. The modified allocation, y_i, has the same utility, and belongs to the interior of the budget set. Therefore, there exists a positive ϵ such that the allocation $(1+\epsilon) \cdot y_i$ belongs to the budget set and has higher utility than x_i. Therefore x is not a competitive equilibrium allocation. Contradiction!

QED

As a consequence of the fact that the utility is measurable with respect to the information of the agents, in equilibrium, "pessimistic" expected utility is equal to normal expected utility. For any prior probabilities over states of nature, $q_i(\omega)$, consistent with the given prior probabilities over observed events, $q_i(A_i^k)$, we have:

$$\sum_{A_i^k \in P_i} q_i(A_i^k) \ \min_{\omega \in A_i^k} u_i^{\omega}\ (x_i(\omega)) = \sum_{\omega \in \Omega} q_i(\omega)\ u_i^{\omega}\ (x_i(\omega)).$$

In states that are not distinguished, agents select different consumption bundles to take advantage of variations in prices.

Theorem 2 *Let* (x, p) *be a competitive equilibrium.*

$\omega' \in P_i(\omega) \Rightarrow p(\omega) \cdot x_i(\omega) \leq p(\omega) \cdot x_i(\omega')$.

Proof. Suppose that for some $\omega' \in P_i(\omega)$, we had $p(\omega) \cdot x_i(\omega) > p(\omega) \cdot x_i(\omega')$. Designate by y_i a modified bundle with consumption of $x_i(\omega')$ in state ω (instead of $x_i(\omega)$). This bundle gives the same utility and allows the agent to retain some rent. There exists a positive ϵ such that $(1 + \epsilon) \cdot y_i$ belongs to the budget set and has higher utility than x_i. Therefore, x is not a competitive equilibrium allocation. Contradiction!

QED

In spite of the "pessimistic preferences", expected utility in equilibrium is still higher in the sense of Pareto than that which is attainable under the classical restriction of equal consumption in states of nature that are not distinguished ("private core" and "Walrasian expectations equilibrium"). Efficiency of exchange is enhanced in a sense that we make precise below.

Theorem 3 *Let* (x, p) *be a Walrasian expectations equilibrium (Radner) of the economy.*

There are Pareto optima of the economy with uncertain delivery, z*, such that* $U_i(z_i) \geq U_i(x_i)$ *for every agent* $i = 1, ..., n$*. There are examples in which the improvement is strict (see sections 3 and 4).*

Proof. Let (x, p) be a Walrasian expectations equilibrium (Radner) of the economy. The allocation x is still feasible in the economy with uncertain delivery.

QED

In general, prices vary across states in some A_i, so theorem 4 suggests that Walrasian expectations equilibria are not competitive equilibria of the economy with uncertain delivery.

Theorem 4 *Let* $Y(\omega) = \{y^i \subset \mathbb{R}^l_+ : u^i(P^i(\omega), y^i) = u^i(P^i(\omega), x^i(\omega))\}$.

If for some ω*, we have:* $p(\omega) \cdot x^i(\omega) > \min_{y^i \in Y(\omega)} p(\omega) \cdot y^i$*, then* x *is not a competitive equilibrium allocation of the economy with uncertain delivery.*

Proof. Assume that (x, p) is a competitive equilibrium, and that there exists some ω such that $p(\omega) \cdot x^i(\omega) > \min_{y^i \in A(\omega)} p(\omega) \cdot y^i$. The modified allocation with y^i instead of $x^i(\omega)$ has the same utility and is rent saving. Therefore, we can multiply this modified allocation by $(1 + \epsilon)$, with $\epsilon > 0$ and obtain an allocation in the budget set that has higher utility. Therefore, x isn't a competitive equilibrium allocation. Contradiction!

QED

Cooperative solutions can also be analyzed. The core of this modified economy may be designated as the "uncertain private core" of the economy with differential information. It is the set of all feasible allocations which are not blocked by any coalition. Although coalitions of agents are formed, information is not shared between them. The transformation of the primitive preferences to pessimistic expected utility is based only on each agent's private information.

A coalition $S \subseteq N$ privately blocks an allocation x if there exists $(y_i)_{i \in S}$ such that: $\sum_{i \in S} y_i \leq \sum_{i \in S} e_i$ and $U_i(y_i) > U_i(x_i)$ for every $i \in S$.

The "uncertain private core" is very similar to a modified private core where measurable utility is required instead of measurable consumption. Actually, measurable utility is not required, but, given an allocation in the uncertain private core, there exists another with equal utility for every agent, having measurable utility and requiring less resources.

Theorem 5 *Let* $x \in Core(\mathcal{E})$.

There exists some $x' \in Core(\mathcal{E})$ *such that,* $\forall i = 1, ..., n$*:*

a) $x'_i \leq x_i$;

b) $U_i(x'_i) = U_i(x_i)$;

c) $u_i(\omega, x'_i)$ *is* P_i*-measurable.*

Proof. If $u_i(\omega, x_i)$ isn't P_i-measurable, we can multiply the $x_i(\omega)$ that have higher utilities in each element of P_i by a factor smaller than 1 to obtain a modified allocation with measurable utility. These higher utilities were not considered in the calculation of expected utility, because only the worst outcome is considered. Therefore, expected utility remains unchanged and this allocation satisfies $x'_i \leq x_i$ and $U_i(x'_i) = U_i(x_i)$.

QED

Evaluated by the (pessimistic) expected utilities, allocations in the "uncertain private core" dominate, in the sense of Pareto, the "private core" (Yannelis, 1991) allocations. The latter are feasible in the economy for uncertain delivery, while the converse is not true. Efficiency of exchange is enhanced while incentive compatibility is preserved, independently of the beliefs regarding the preferences of the other agents.

References

Arrow, K.J. (1953), "The Role of Securities in the Optimal Allocation of Risk-Bearing", *Econometrie*, translated and reprinted in 1964, *Review of Economic Studies*, Vol. 31, pp. 91-96.

Debreu, G. (1959), "Theory of Value", Wiley, New York.

Radner, R. (1968), "Competitive Equilibrium under Uncertainty", *Econometrica*, 36, 1, pp. 31-58.

von Neumann, J. and O. Morgenstern (1944), "Theory of Games and Economic Behavior", Princeton University Press, Princeton.

Yannelis, N.C. (1991), "The Core of an Economy with Differential Information", *Economic Theory*, 1, pp. 183-198.