Orthogonality Constraints on Vectors in k Dimensions

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The equations

- Let $\{\mathbf{x}_i\}$ be n (unknown) vectors in $\mathbf{R}^k$ (real Euclidean vector space (Hilbert Space) of k dimensions).
- Investigate solutions to the equations

  $\{ \mathbf{x}_i.\mathbf{x}_j = 0$, for all (i,j) in E }, where E is a specified subset of all 2^n integer pairs
- The equations are specified by k, n and E. There are e=|E| equations.

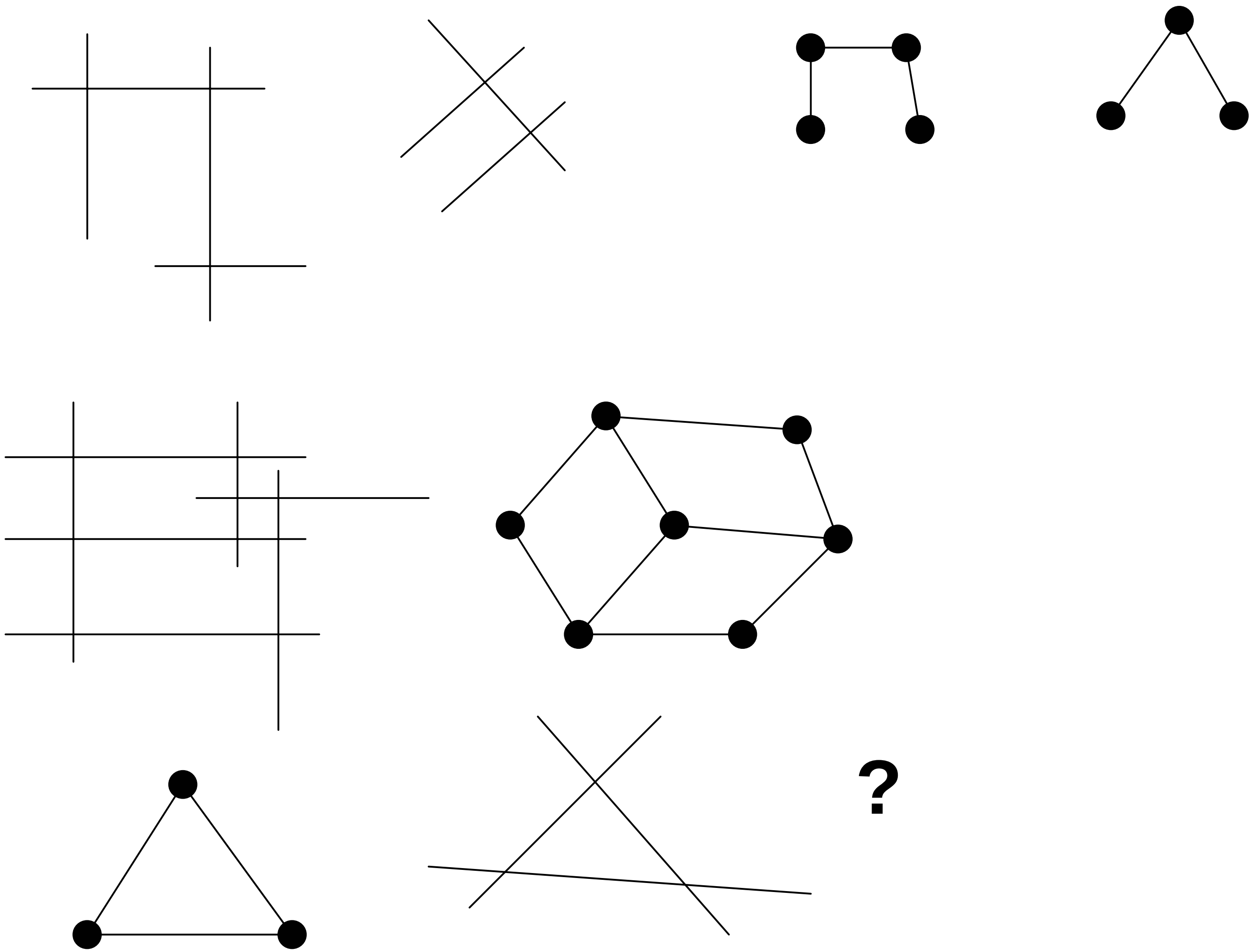
Motivation

- For k = 2 these equations arise in solving constrained sketches – but are trivial to solve.
- For k = 3 these equations arise when solving dimensional equations for modifying Solid Models – the motivation for this investigation.
- For k > 3 - mathematical interest.

Properties of the equations

- Equations are locally homogeneous – if $\{\mathbf{p}_i\}$ is a solution then $\{a_i\mathbf{p}_i\}$ is also a solution. Could be removed by fixing $|\mathbf{p}_i| = 1$.
- Equations are invariant under k(k-1)/2 orthogonal transformations.
- freedoms = kn-e-n-k(k-1)/2 = nd-e-d(d+1)/2
- Same as “points” in d = k-1 dimensions.

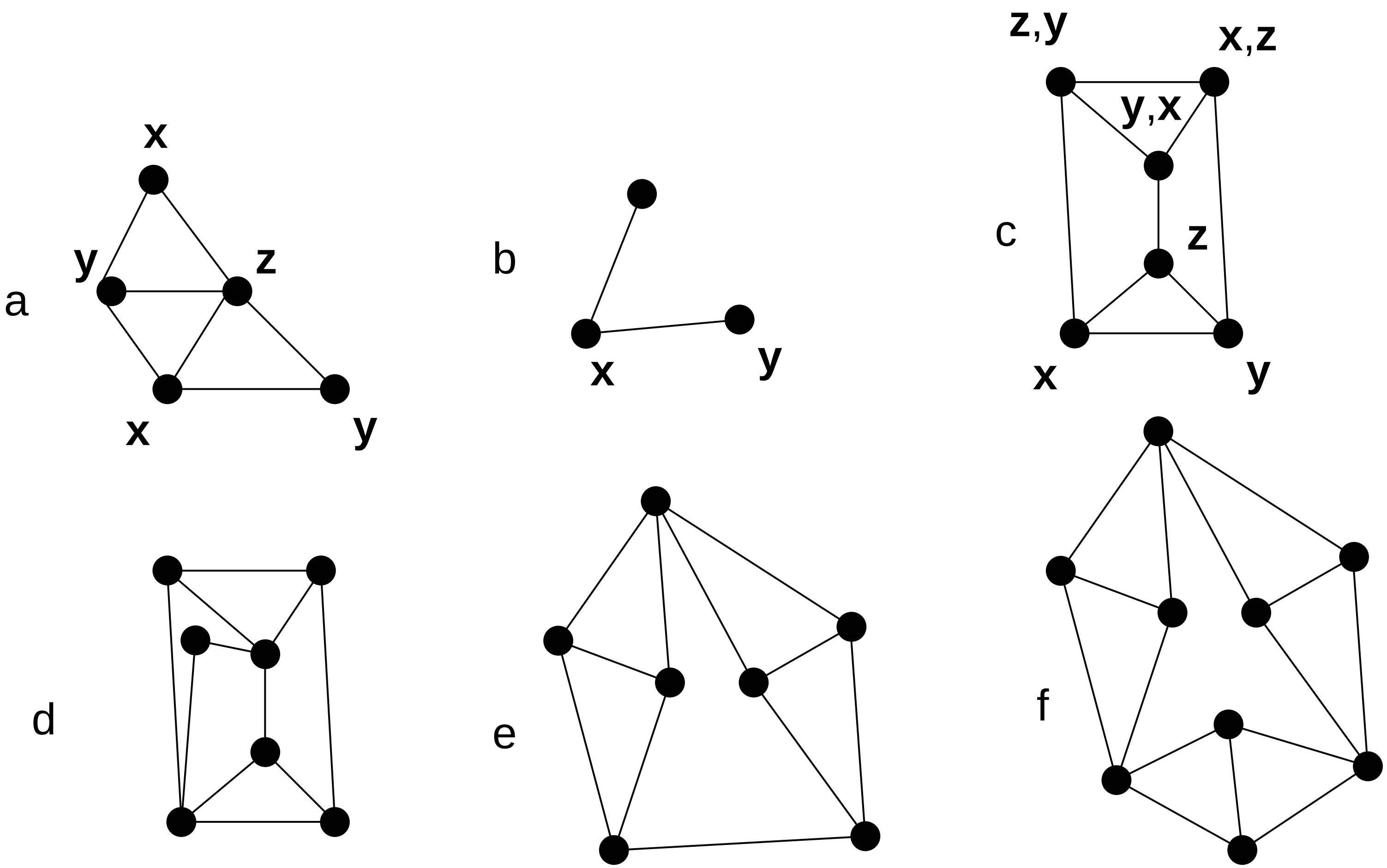
k=2



k=2

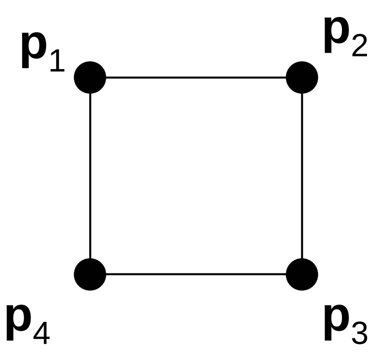
- Equations have a solution if and only if all cycles are even. If the graph is connected the equations have a unique *rigid* solution, which is a *basis* solution. Disconnected graphs also have *non-basis* solutions which are *not rigid*. Every *rigid* solution is a *basis* solution.
- A graph is 2-colorable if and only if all its cycles are even. Thus, the equations have a solution if and only if the graph is 2-colorable. The solution is rigid if and only if the graph is connected. Both conditions can be tested in O(n+e) time.

k=3



Some graphs showing simple constraint systems for
k=3

k=3



- if $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4$ are in a 4-cycle then either $\mathbf{p}_1$ is parallel (equal) to $\mathbf{p}_3$ or $\mathbf{p}_2$ is parallel (equal) to $\mathbf{p}_4$.
- a has a unique rigid solution
- b has non-rigid solutions
- c has 2 rigid solutions
- d has a rigid solution and non-rigid solutions
- e has no solutions
- f has non-rigid solutions

Basis solutions for k>2

- For any k, a basis solution has all vectors in one basis for $\mathbf{R}^k$ – for example **(1,0,0)**, **(0,1,0)**, or **(0,0,1)** for k=3)
- The equations have a basis zero if and only if the graph is k-colourable – proof: map the basis vector $\mathbf{b}_i$ to the colour i.
- k-colourability is NP-complete, thus existence of basis solutions in NP complete – what about existence of any solutions (existence of embedding)?

Rigidity of basis solutions

Theorem: A basis solution is rigid if and only if the corresponding graph colouring has no vertex separation sets which use fewer than k-1 colours. Note: true for k=2.

Proof: “only if” is easy (e.g. in k=3 a separation set with only 1 vector direction allows “rotation” around this direction).

“if” sketch proof.

Consider the k(k-1)/2 subgraphs formed from the edges which connect only vertices which have a specific pair of colours.

Each of these subgraphs is connected. Otherwise the vertices which use the remaining k-2 colours would separate the graph.

Let each colour class pair have n_{ij} vertices and e_{ij} edges. Then $e_{ij} >= n_{ij}$ - 1 and summing over all pairs gives e>= (k-1)n-k(k-1)/2 as required for rigidity.

Can show that the Jacobean matrix has rank (k-1)n-k(k-1)/2 which proves infinitesimal rigidity and thus rigidity.

Non-basis solutions

- Are all rigid solutions basis solutions? Yes for k=2 and yes for small k=3 examples. No in general.
- If the equations have a non-basis solution do they also have a basis solution – “embeddable” implies “basis embeddable”? Yes for k=2 and yes for small k=3 examples. No in general.
- Look for *minimal* counter-examples – for k=3, these must be graphs without 4-cycles. They are hard to find – e.g. a Laman graph with no 4-cycles.

Hidden Variables in Quantum Mechanics

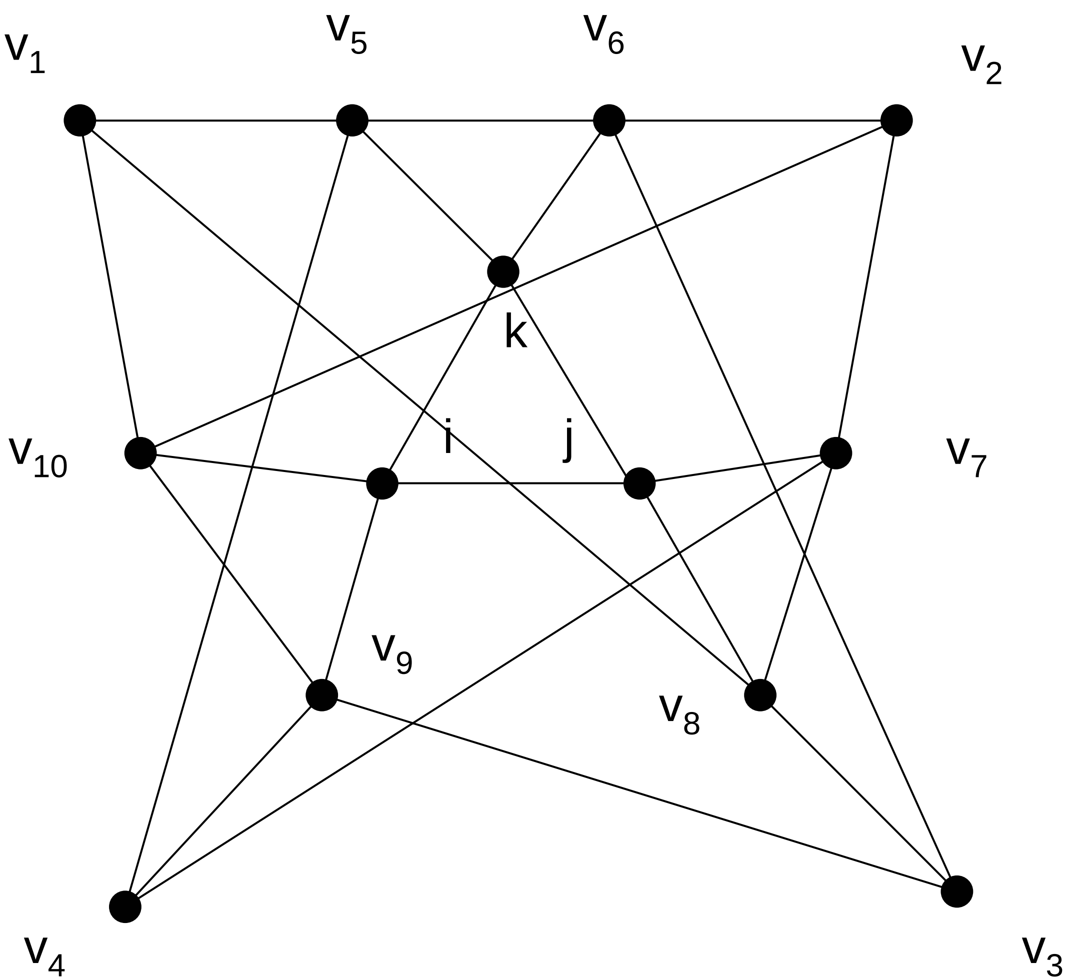
- Quantum Mechanics associates physical observables with operators in a Hilbert Space and determines only probabilities for measured values.
- A theory with Hidden Variables seeks to replace the probabilities by assigning definite values to the vectors. In the simplest cases there are just two possible “values” red and green.
- It can be shown that this is impossible if there is any set of vectors (KS) in a k-dimensional Hilbert Space which cannot be assigned the colors red and green (QM-coloured) such that:
  - In every subset of KS comprising k mutually orthogonal vectors (a basis), exactly one vector is assigned the colour red (and the rest green)
  - No two vectors with the colour red are orthogonal.

Kocken and Specher (1967) found a set KS with 117 vectors for k=3. Peres(1991) subsequently found a set with 33 vectors for k=3 and a set with 24 vectors for k=4.

Hidden variables and orthogonal constraints

- If the set KS cannot be QM-coloured then it certainly cannot be k-coloured, otherwise one of the k colours could be assigned red and the others green to give a QM-colouring.
- Cannot colour all the vectors of a k-dimensional Hilbert space with k colours (so that orthogonal vectors have different colours).
- This proves there are equations which have only non-basis zeros (for $k=3$) and in fact for all $k>2$.
- Based on this work we have found smaller sets which cannot be k-coloured and equations which have rigid non-basis solutions.

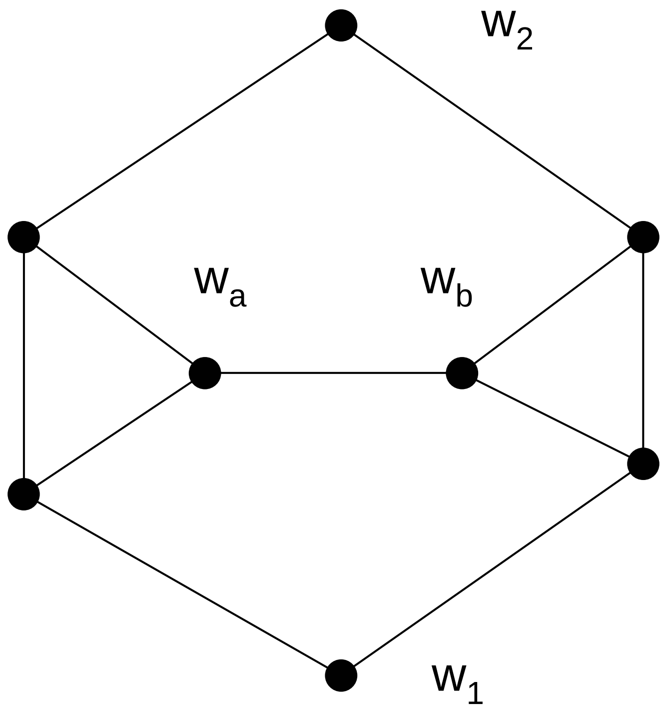
A graph which gives only non-basis solutions (k=3)



The graph P_{13} which gives only non-basis solutions

P_{13} cannot be 3-coloured

- P_{13} cannot be 3-coloured. Easy to show by trial and error or by using the subgraph



In any 3-colouring, w_1 and w_2 have different colours because if they have the same colour then w_a and w_b also have this colour.

Every pair from the vertices v_1, v_2, v_3 and v_4 thus have different colours which is impossible with 3 colours (cannot 3-colour K_4)

Solutions to P_{13}

- P_{13} has non-basis solutions as follows:
- Let i, j, and k be (1,0,0),(0,1,0) and (0,0,1)
- Let $v_1 = (a,b,c)$
- Then $v_5=(-b,a,0)$;$v_6=(a,b,0)$;$v_{10}=(0,c,-b)$;$v_9=(0,b,c)$;

$v_8=(c,0,-a)$;$v_7=(a,0,c)$

- The constraints on the three vectors v_2,v_3 and v_4 imply 3 co-planarity conditions:
- $a(b^2-c^2)=0$; $b(c^2-a^2)=0$; $c(a^2-b^2)=0$. These are satisfied by $a^2=b^2=c^2$.
- P_{13} has 4 distinct, rigid non-basis solutions.
- Solution vectors are (1,0,0), (0,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,0), (0,1,-1), (1,0,-1), (1,-1,0), (-1,1,1), (1,-1,1), (1,1,-1), (1,1,1)
- $P_{12} = P_{13} \backslash v_1$ has 2 distinct 3-colorings. It has 2 basis solutions and 4 non-basis solutions which are all rigid.

Remaining questions

- Is P_{12} the smallest equation set with rigid non-basis solutions for k=3?
- Is P_{13} the smallest equation set with only non-basis solutions for k=3?
- Study non-basis solutions:
  - Combinatorial condition for non-basis solutions?
  - Combinatorial condition for rigidity of non-basis solutions?
  - Is embedability of orthogonal constraints NP-complete for k>2?