Geometric Factorization and Theorem Completion Using Bracket Algebras

HONGBO LI

- Research Background: Geometric Division
- Algebra of Invariants and Covariants
- Null Bracket Algebra and Applications
- Conclusion

1. Research Background: Geometric Division

Geometric computation:

1. Algebraic representation;
2. Algebraic Computation;
3. Geometric interpretation.

When (Cartesian) coordinates are used, geometric relations are represented by polynomials of coordinates.

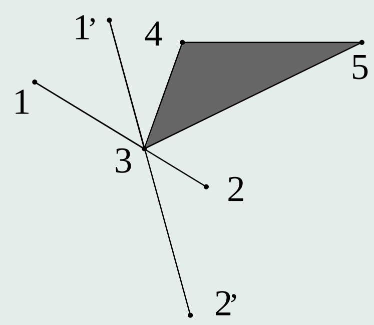
When applied to geometric problems, algebraic results are usually inexplicable geometrically.

A Trivial Example

In the projective plane let $\mathbf{3} = \mathbf{12} \cap \mathbf{1'2'}$. Let $\mathbf{i} = (x_i, y_i, z_i)$. Compute

$$[\mathbf{345}] = \begin{vmatrix} x_3 & x_4 & x_5 \\ y_3 & y_4 & y_5 \\ z_3 & z_4 & z_5 \end{vmatrix}.$$

In affine geometry, $[\mathbf{345}] = 2\, S_{\mathbf{345}}$. It equals 0 iff $\mathbf{3}, \mathbf{4}, \mathbf{5}$ are collinear.



Further ask:

If $\mathbf{3}$ is not strictly on lines $\mathbf{12}$ and $\mathbf{1'2'}$, how does $[\mathbf{345}]$ vary?

In coordinates:

.

```
> f1:=det([[x1,y1,z1],[x2,y2,z2],[x3,y3,z3]])-a:
> f2:=det([[x1p,y1p,z1p],[x2p,y2p,z2p],[x3,y3,z3]])-b:
> g:=det([[x4,y4,z4],[x5,y5,z5],[x3,y3,z3]]):
> h:=solve({f1,f2},{x3,y3}):
> fin:=simplify(subs(h,g)):
> target:=collect(fin,[a,b]);
```

$$
\begin{aligned}
\mathit{target} := & -\frac{(-z4\,y5\,x1p\,z2p + y4\,z5\,x1p\,z2p + x5\,z4\,y1p\,z2p - x5\,z4\,z1p\,y2p - x4\,z5\,y1p\,z2p + x4\,z5\,z1p\,y2p - y4\,z5\,x2p\,z1p + z4\,y5\,x2p\,z1p)\,a}{-x1p\,z2p\,y1\,z2 + x1p\,z2p\,z1\,y2 + x2p\,z1p\,y1\,z2 - x2p\,z1p\,z1\,y2 + y1p\,z2p\,x1\,z2 - y1p\,z2p\,x2\,z1 - z1p\,y2p\,x1\,z2 + z1p\,y2p\,x2\,z1} \\
& - \frac{(-x4\,z5\,z1\,y2 - y4\,z5\,x1\,z2 + x4\,z5\,y1\,z2 - x5\,z4\,y1\,z2 + x5\,z4\,z1\,y2 + y4\,z5\,x2\,z1 + z4\,y5\,x1\,z2 - z4\,y5\,x2\,z1)\,b}{-x1p\,z2p\,y1\,z2 + x1p\,z2p\,z1\,y2 + x2p\,z1p\,y1\,z2 - x2p\,z1p\,z1\,y2 + y1p\,z2p\,x1\,z2 - y1p\,z2p\,x2\,z1 - z1p\,y2p\,x1\,z2 + z1p\,y2p\,x2\,z1} \\
& - \frac{1}{-x1p\,z2p\,y1\,z2 + x1p\,z2p\,z1\,y2 + x2p\,z1p\,y1\,z2 - x2p\,z1p\,z1\,y2 + y1p\,z2p\,x1\,z2 - y1p\,z2p\,x2\,z1 - z1p\,y2p\,x1\,z2 + z1p\,y2p\,x2\,z1} \\
& (x4\,y5\,z3\,x1p\,z2p\,y1\,z2 - x4\,y5\,z3\,x1p\,z2p\,z1\,y2 - x4\,y5\,z3\,x2p\,z1p\,y1\,z2 + x4\,y5\,z3\,x2p\,z1p\,z1\,y2 - x4\,y5\,z3\,y1p\,z2p\,x1\,z2 \\
& + x4\,y5\,z3\,y1p\,z2p\,x2\,z1 + x4\,y5\,z3\,z1p\,y2p\,x1\,z2 - x4\,y5\,z3\,z1p\,y2p\,x2\,z1 - x4\,z5\,x1p\,y2p\,z3\,y1\,z2 + x4\,z5\,x1p\,y2p\,z3\,z1\,y2 \\
& + x4\,z5\,x2p\,y1p\,z3\,y1\,z2 - x4\,z5\,x2p\,y1p\,z3\,z1\,y2 + x4\,z5\,y1p\,z2p\,x1\,y2\,z3 - x4\,z5\,y1p\,z2p\,x2\,y1\,z3 - x4\,z5\,z1p\,y2p\,x1\,y2\,z3 \\
& + x4\,z5\,z1p\,y2p\,x2\,y1\,z3 - x5\,y4\,z3\,x1p\,z2p\,y1\,z2 + x5\,y4\,z3\,x1p\,z2p\,z1\,y2 + x5\,y4\,z3\,x2p\,z1p\,y1\,z2 - x5\,y4\,z3\,x2p\,z1p\,z1\,y2 \\
& + x5\,y4\,z3\,y1p\,z2p\,x1\,z2 - x5\,y4\,z3\,y1p\,z2p\,x2\,z1 - x5\,y4\,z3\,z1p\,y2p\,x1\,z2 + x5\,y4\,z3\,z1p\,y2p\,x2\,z1 + x5\,z4\,x1p\,y2p\,z3\,y1\,z2 \\
& - x5\,z4\,x1p\,y2p\,z3\,z1\,y2 - x5\,z4\,x2p\,y1p\,z3\,y1\,z2 + x5\,z4\,x2p\,y1p\,z3\,z1\,y2 - x5\,z4\,y1p\,z2p\,x1\,y2\,z3 + x5\,z4\,y1p\,z2p\,x2\,y1\,z3 \\
& + x5\,z4\,z1p\,y2p\,x1\,y2\,z3 - x5\,z4\,z1p\,y2p\,x2\,y1\,z3 + y4\,z5\,x1p\,y2p\,z3\,x1\,z2 - y4\,z5\,x1p\,y2p\,z3\,x2\,z1 - y4\,z5\,x1p\,z2p\,x1\,y2\,z3 \\
& + y4\,z5\,x1p\,z2p\,x2\,y1\,z3 - y4\,z5\,x2p\,y1p\,z3\,x1\,z2 + y4\,z5\,x2p\,y1p\,z3\,x2\,z1 + y4\,z5\,x2p\,z1p\,x1\,y2\,z3 - y4\,z5\,x2p\,z1p\,x2\,y1\,z3 \\
& - z4\,y5\,x1p\,y2p\,z3\,x1\,z2 + z4\,y5\,x1p\,y2p\,z3\,x2\,z1 + z4\,y5\,x1p\,z2p\,x1\,y2\,z3 - z4\,y5\,x1p\,z2p\,x2\,y1\,z3 + z4\,y5\,x2p\,y1p\,z3\,x1\,z2 \\
& - z4\,y5\,x2p\,y1p\,z3\,x2\,z1 - z4\,y5\,x2p\,z1p\,x1\,y2\,z3 + z4\,y5\,x2p\,z1p\,x2\,y1\,z3)
\end{aligned}
$$

Is there any pro-geometric explanation?

By GREAT efforts, the expression is recognized in Grassmann-Cayley algebra as

$$\frac{-a\,\mathbf{1'2'}\wedge\mathbf{45}\wedge\mathbf{xy}+b\,\mathbf{12}\wedge\mathbf{45}\wedge\mathbf{xy}+z_3\,\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{45}}{\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{xy}}.$$

Or more explicitly:

$$\frac{-[\mathbf{123}]\,\mathbf{1'2'}\wedge\mathbf{45}\wedge\mathbf{xy}+[\mathbf{1'2'3}]\,\mathbf{12}\wedge\mathbf{45}\wedge\mathbf{xy}+[\mathbf{3xy}]\,\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{45}}{\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{xy}}.$$

Here $\mathbf{x}, \mathbf{y}$ are the x- and y- coordinate basis vectors.

In Grassmann-Cayley algebra, substituting $\mathbf{3} = \mathbf{12} \wedge \mathbf{1'2'}$ into $[\mathbf{345}]$, one obtains

$$\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{45} = [\mathbf{124}][\mathbf{1'2'5}] - [\mathbf{125}][\mathbf{1'2'4}].$$

$$\mathbf{12}, \mathbf{1'2'}, \mathbf{45} \text{ concur iff } \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{45} = 0.$$

In affine geometry, let $\mathbf{0} = \mathbf{12} \cap \mathbf{1'2'}$, then

$$\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{xy} = \frac{S_{\mathbf{0xy}}}{S_{\mathbf{11'22'}}}.$$

Formulation of the problem:

Do pseudodivision of $[\mathbf{345}]$ by $[\mathbf{123}]$ and $[\mathbf{1'2'3}]$ (with $\mathbf{3}$ as leading vector variable), find the pseudo-quotients and pseudo-remainder:

$$-\frac{\mathbf{1'2'} \wedge \mathbf{45} \wedge \mathbf{xy}}{\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{xy}}, \quad \frac{\mathbf{12} \wedge \mathbf{45} \wedge \mathbf{xy}}{\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{xy}}, \quad \frac{[\mathbf{3xy}]\, \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{45}}{\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{xy}}.$$

Observation

- The devision is never unique: it depends on the choice of $\mathbf{x}, \mathbf{y}$.
- If choosing $\mathbf{xy} = \mathbf{45}$ then nothing is done.
- If choosing $\mathbf{x} \in \{\mathbf{1}', \mathbf{2}'\}$ and $\mathbf{y} \in \{\mathbf{1}, \mathbf{2}\}$ then the result has lowest degrees.

  E.g., if $\mathbf{x} = \mathbf{1}'$ and $\mathbf{y} = \mathbf{1}$ then

$$[\mathbf{345}] = \frac{[\mathbf{451}']}{[\mathbf{121}']}[\mathbf{123}] + \frac{[\mathbf{145}]}{[\mathbf{11}'\mathbf{2}']}[\mathbf{1}'\mathbf{2}'\mathbf{3}] + \frac{[\mathbf{131}']\,\mathbf{12} \wedge \mathbf{1}'\mathbf{2}' \wedge \mathbf{45}}{[\mathbf{121}'][\mathbf{11}'\mathbf{2}']}.$$

Can we find an easier approach based on brackets and wedge products?

Geometric Factorization

A geometric theorem:

hypotheses $\{h_1 = 0, h_2 = 0, \cdots, h_m = 0\}$ and conclusion $c = 0$.

Now discard several h's, say $\{h_1, h_2, \cdots, h_l\}$, from the hypotheses.

Find how the expression c is related to the discarded hypotheses quantitatively and *geometrically*.

E.g., if c can be written in the following form

$$c^r = \lambda_1 h_1 + \lambda_2 h_2 + \cdots + \lambda_l h_l$$

after reduction by $\{h_{l+1}, h_{l+2}, \cdots, h_m\}$, and the λ's have immediate geometric interpretation, then the factorization is "*geometric*".

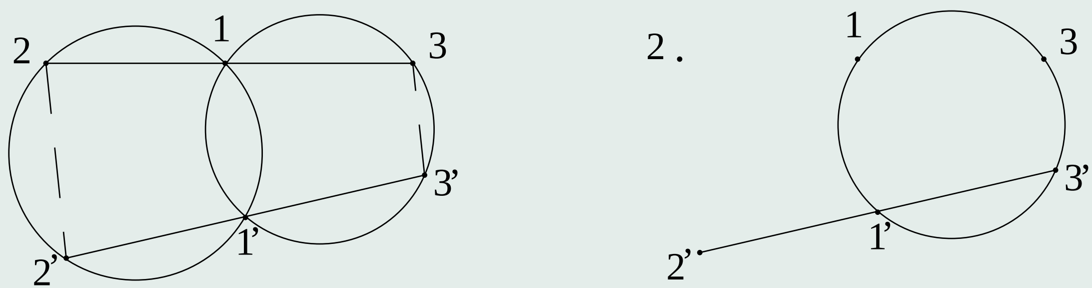
Geometric Theorem Completion

The *completion* of the original theorem with basic constraints $\{h_{l+1} = 0, h_{l+2} = 0, \cdots, h_m = 0\}$, refers to the finding of the

necessary and sufficient geometric conditions

for the conclusion $c = 0$ to be true.

E.g. conclusion: $\mathbf{22}' \,//\, \mathbf{33}'$.



Keep: $\mathbf{131'3'}$ be a circle, $\mathbf{1'2'3'}$ be a line.

Discard: $\mathbf{121'2'}$ be a circle, $\mathbf{123}$ be a line.

Computation result using null bracket algebra by eliminating $\mathbf{3'}$:

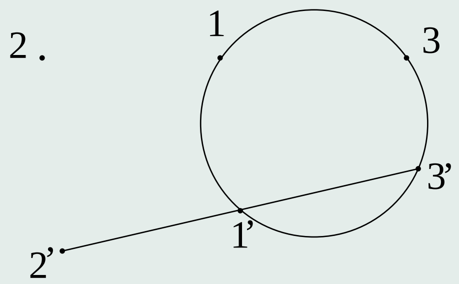
$$\frac{[\mathbf{e22'e33'}]}{\mathbf{e}\cdot\mathbf{3'}} = \frac{[\mathbf{e31'2'}][\mathbf{e311'2'2}]}{\mathbf{1'}\cdot\mathbf{2'}\,[\mathbf{e131'}]}.$$

Geometrically:

$$\mathbf{22'}\times\mathbf{33'} = \frac{2\,S_{\mathbf{21'2'}}S_{\mathbf{31'2'}}\sin(\angle(\mathbf{31},\mathbf{11'})+\angle(\mathbf{1'2'},\mathbf{2'2}))}{d^2_{\mathbf{1'2'}}}.$$

This is an extension of the theorem.

Partial explanation:



$\mathbf{22'}||\mathbf{33'}$ iff

$$\angle(\mathbf{31}, \mathbf{11'}) + \angle(\mathbf{1'2'}, \mathbf{2'2})) = 0,$$

i.e.,

$$\angle\mathbf{1'3'3} = \pi - \angle\mathbf{1'2'2}.$$

Obvious.

$[\mathbf{e311'2'2}]$ has the following *rational expansion*:

$$\frac{1}{2}[\mathbf{e311'2'2}] = \frac{\mathbf{1}\cdot\mathbf{3}[\mathbf{e232'}][\mathbf{121'2'}] - \mathbf{2}\cdot\mathbf{2'}[\mathbf{e123}][\mathbf{131'2'}]}{[\mathbf{1232'}]}.$$

The discarded hypotheses are both included in the expansion:

$$[\mathbf{121'2'}] = 0 \text{ iff } \mathbf{121'2'} \text{ be a circle,}$$
$$[\mathbf{e123}] = 0 \text{ iff } \mathbf{123} \text{ be a line.}$$

This is the geometric factorization.

2. Algebra of Geometric Covariants and Invariants

Hope: more intrinsic algebraic system may help maintaining more geometry.

Classical Invariant Theory: Invariance under $\mathcal{GL}_n(\mathcal{K})$.

- Algebra of Covariants: Grassmann-Cayley algebra. Outer product, meet product: Denoted by juxtaposition and "$\wedge$".

  E.g., the intersection of lines $\mathbf{12}$, $\mathbf{1'2'}$ is

  $$\mathbf{12} \wedge \mathbf{1'2'} = [\mathbf{122'}]\mathbf{1'} - [\mathbf{121'}]\mathbf{2'} = [\mathbf{11'2'}]\mathbf{2} - [\mathbf{21'2'}]\mathbf{1}.$$ (shuffle formula)

- Algebra of Invariants: Bracket Algebra. An nD bracket algebra generated by a sequence of $m > n+1$ symbols of vectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$, is

$$\frac{\text{Polynomial ring generated by all subsequences of length } n}{\text{Ideal generated by Grassmann-Plücker syzygies}}$$

GP:

$$\sum_{k=1}^{n+1} [\mathbf{i}_1\mathbf{i}_2 \cdots \mathbf{i}_{n-1}\mathbf{j}_k][\mathbf{j}_1\mathbf{j}_2 \cdots \check{\mathbf{j}}_k \cdots \mathbf{j}_{n+1}],$$

i.e., $\mathbf{i}_1\mathbf{i}_2 \cdots \mathbf{i}_{n-1} \wedge \mathbf{j}_1\mathbf{j}_2 \cdots \mathbf{j}_{n+1}$ applied with shuffle formula.

In affine geometry, $[\mathbf{123}]$ is the area of triangle $\mathbf{123}$.

E.g., for 2D projective geometry, the 3D bracket algebra generated by 5 points $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}$ is the bracket polynomials with $C_5^3 = 10$ indeterminates

$$[\mathbf{123}], [\mathbf{124}], [\mathbf{125}], \ldots, [\mathbf{345}].$$

The 10 brackets are not algebraically independent. They satisfy 5 algebraic relations:

$$\begin{aligned}
[\mathbf{123}][\mathbf{145}] - [\mathbf{124}][\mathbf{135}] + [\mathbf{125}][\mathbf{134}] &= 0, \\
[\mathbf{123}][\mathbf{245}] - [\mathbf{124}][\mathbf{235}] + [\mathbf{125}][\mathbf{234}] &= 0, \\
[\mathbf{123}][\mathbf{345}] - [\mathbf{134}][\mathbf{235}] + [\mathbf{135}][\mathbf{234}] &= 0, \\
[\mathbf{124}][\mathbf{345}] - [\mathbf{134}][\mathbf{245}] + [\mathbf{145}][\mathbf{234}] &= 0, \\
[\mathbf{125}][\mathbf{345}] - [\mathbf{135}][\mathbf{245}] + [\mathbf{145}][\mathbf{235}] &= 0.
\end{aligned}$$

Among the 5 syzygies, only 3 are algebraically independent, e.g., the first 3. They form a bracket basis of the syzygies.

Invariants vs. Coordinates

$$\mathbf{1} = (x_1, y_1, z_1) = x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3, \quad [\mathbf{123}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

$$[\mathbf{123}][\mathbf{145}] = \text{factorization of } p, \quad [\mathbf{123}][\mathbf{145}] + [\mathbf{124}][\mathbf{135}] = q.$$

$$\begin{aligned}
p = {} & x2*z1*y3*x1*y4*z5 - x2*z1*y3*x1*z4*y5 - x2*z1*y3*x4*y1*z5 + x2*z1^2*y3*x4*y5 \\
& -x1*y2*z3*x4*y1*z5 + x1*y2*z3*x4*z1*y5 + x1*y2*z3*x5*y1*z4 - x1*y2*z3*x5*z1*y4 \\
& -x3*z1*y2*x1*y4*z5 + x3*z1*y2*x1*z4*y5 + x3*z1*y2*x4*y1*z5 - x3*z1^2*y2*x4*y5 \\
& +x3*z1^2*y2*x5*y4 + x1^2*y2*z3*y4*z5 - x1^2*y2*z3*z4*y5 - x1*z2*y3*x5*y1*z4 \\
& +x1*z2*y3*x5*z1*y4 - x2*y1*z3*x1*y4*z5 + x2*y1*z3*x1*z4*y5 + x2*y1^2*z3*x4*z5 \\
& -x2*y1*z3*x4*z1*y5 - x2*y1^2*z3*x5*z4 + x2*y1*z3*x5*z1*y4 + x2*z1*y3*x5*y1*z4 \\
& -x2*z1^2*y3*x5*y4 + x3*y1*z2*x1*y4*z5 - x3*y1*z2*x1*z4*y5 - x3*y1^2*z2*x4*z5 \\
& +x3*y1*z2*x4*z1*y5 + x3*y1^2*z2*x5*z4 - x3*y1*z2*x5*z1*y4 - x1^2*z2*y3*y4*z5 \\
& +x1^2*z2*y3*z4*y5 + x1*z2*y3*x4*y1*z5 - x1*z2*y3*x4*z1*y5 - x3*z1*y2*x5*y1*z4
\end{aligned}$$

$$\begin{aligned}
q = {} & -x1*y2*z4*x3*y1*z5 - x1*y2*z4*x5*z1*y3 + x1^2*z2*y4*z3*y5 - x1*z2*y4*x3*z1*y5 \\
& -x1*z2*y4*x5*y1*z3 - x2*y1*z4*x1*y3*z5 + x2*y1^2*z4*x3*z5 - x2*y1*z4*x3*z1*y5 \\
& +2*x2*z1*y3*x1*y4*z5 - x2*z1*y3*x1*z4*y5 - x2*z1*y3*x4*y1*z5 + x2*z1^2*y3*x4*y5 \\
& -x1*y2*z3*x4*y1*z5 + 2*x1*y2*z3*x4*z1*y5 + 2*x1*y2*z3*x5*y1*z4 - x1*y2*z3*x5*z1*y4 \\
& -x3*z1*y2*x1*y4*z5 + 2*x3*z1*y2*x1*z4*y5 + 2*x3*z1*y2*x4*y1*z5 - 2*x3*z1^2*y2*x4*y5 \\
& +x3*z1^2*y2*x5*y4 + x1^2*y2*z3*y4*z5 - 2*x1^2*y2*z3*z4*y5 - x1*z2*y3*x5*y1*z4 - x2*z1*y4*x3*y1*z5 \\
& +2*x1*z2*y3*x5*z1*y4 - x2*y1*z3*x1*y4*z5 + 2*x2*y1*z3*x1*z4*y5 + x2*y1^2*z3*x4*z5 \\
& -x2*y1*z3*x4*z1*y5 - 2*x2*y1^2*z3*x5*z4 + 2*x2*y1*z3*x5*z1*y4 + 2*x2*z1*y3*x5*y1*z4 \\
& -2*x2*z1^2*y3*x5*y4 + 2*x3*y1*z2*x1*y4*z5 - x3*y1*z2*x1*z4*y5 - 2*x3*y1^2*z2*x4*z5 \\
& +2*x3*y1*z2*x4*z1*y5 + x3*y1^2*z2*x5*z4 - x3*y1*z2*x5*z1*y4 - x4*z1*y2*x1*y3*z5 \\
& -x4*z1*y2*x5*y1*z3 + x4*z1^2*y2*x5*y3 - 2*x1^2*z2*y3*y4*z5 + x1^2*z2*y3*z4*y5 + x2*z1^2*y4*x3*y5 \\
& +2*x1*z2*y3*x4*y1*z5 - x1*z2*y3*x4*z1*y5 - x3*z1*y2*x5*y1*z4 + x1^2*y2*z4*y3*z5 \\
& -x4*y1*z2*x1*z3*y5 + x4*y1^2*z2*x5*z3 - x4*y1*z2*x5*z1*y3 - x2*z1*y4*x1*z3*y5
\end{aligned}$$

On the other hand, using only a fixed bracket basis reduces brackets to coordinates, thus is equivalent to using coordinates.

E.g., in the projective plane let $\mathbf{1}, \mathbf{2}, \mathbf{3}$ be a basis (linear independent).

Cramer's rule provides coordinate representations of points:

$$[\mathbf{123}]\mathbf{4} = [\mathbf{234}]\mathbf{1} - [\mathbf{134}]\mathbf{2} + [\mathbf{124}]\mathbf{3}.$$

A bracket basis syzygy is of the form

$$[\mathbf{123}][\mathbf{145}] = [\mathbf{124}][\mathbf{135}] - [\mathbf{125}][\mathbf{134}], \text{ or}$$
$$[\mathbf{123}][\mathbf{456}] = [\mathbf{124}][\mathbf{356}] - [\mathbf{125}][\mathbf{346}] + [\mathbf{126}][\mathbf{345}].$$

They are equivalent to substituting the Cramer's rule of $\mathbf{4}$ into $[\mathbf{145}], [\mathbf{456}]$.

Special Features of Bracket Algebra

- Straightening: Reduce the polynomial into a unique normal form wrt a basis of the $\mathcal{Z}$-module of polynomials.

  However, straightening “explodes” a bracket monomial into a bracket polynomial of many terms. It cannot keep geometric meaning.

- Contraction: Reduce the number of terms of a bracket polynomial.

  Important both in algebraic computation and geometric interpretation.

- Cayley factorization: Translation from the algebra of invariants to the algebra of covariants.

E.g., do pseudodivision to $[\mathbf{345}]$ by $[\mathbf{123}]$ and $[\mathbf{1'2'3}]$, with $\mathbf{3}$ as vector indeterminate.

The pseudo-remainder should contain a factor $\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{45}$.

(1)

$$x[\mathbf{345}] = y[\mathbf{123}] + r$$

$$\downarrow$$

$$[\mathbf{12U}_1][\mathbf{345}] = [\mathbf{45U}_1][\mathbf{123}] - [\mathbf{124}][\mathbf{35U}_1] + [\mathbf{125}][\mathbf{34U}_1].$$

(2)

$$x'[\mathbf{34U}_1] = y'[\mathbf{1'2'3}] + r'$$

$$\downarrow$$

$$[\mathbf{1'2'U}_2][\mathbf{34U}_1] = -[\mathbf{4U}_2\mathbf{U}_1][\mathbf{1'2'3}] + [\mathbf{1'2'U}_1][\mathbf{34U}_2] + [\mathbf{1'2'4}][\mathbf{3U}_2\mathbf{U}_1].$$

Similarly,

$$[\mathbf{1'2'U}_2][\mathbf{35U}_1] = -[\mathbf{5U}_2\mathbf{U}_1][\mathbf{1'2'3}] + [\mathbf{1'2'U}_1][\mathbf{35U}_2] + [\mathbf{1'2'5}][\mathbf{3U}_2\mathbf{U}_1].$$

(3)

$$
\begin{aligned}
[\mathbf{12}\mathrm{U}_1][\mathbf{1'2'}\mathrm{U}_2]\underline{[\mathbf{345}]} \;=\; & [\mathbf{45}\mathrm{U}_1][\mathbf{1'2'}\mathrm{U}_2]\underline{[\mathbf{123}]} \\
& +([\mathbf{124}][\mathbf{5}\mathrm{U}_2\mathrm{U}_1]-[\mathbf{125}][\mathbf{4}\mathrm{U}_2\mathrm{U}_1])\underline{[\mathbf{1'2'3}]} \\
& +(-[\mathbf{124}][\mathbf{1'2'5}]+[\mathbf{125}][\mathbf{1'2'4}])[\mathbf{3}\mathrm{U}_2\mathrm{U}_1] \\
& +(-[\mathbf{124}][\mathbf{35}\mathrm{U}_2]+[\mathbf{125}][\mathbf{34}\mathrm{U}_2])[\mathbf{1'2'}\mathrm{U}_1] \\
\\
= \; & [\mathbf{45}\mathrm{U}_1][\mathbf{1'2'}\mathrm{U}_2]\underline{[\mathbf{123}]} \\
& -\mathbf{12}\wedge\mathbf{45}\wedge\mathrm{U}_2\mathrm{U}_1\;\underline{[\mathbf{1'2'3}]} \\
& -\underline{\mathbf{12}\wedge\mathbf{1'2'}\wedge\mathbf{45}}\;[\mathbf{3}\mathrm{U}_2\mathrm{U}_1] \\
& -\mathbf{12}\wedge\mathbf{45}\wedge\mathbf{3}\mathrm{U}_2\;[\mathbf{1'2'}\mathrm{U}_1]
\end{aligned}
$$

The division is not yet finished: $\mathbf{3}$ is till not completely isolated in $[\mathbf{35}\mathrm{U}_2]$ and $[\mathbf{34}\mathrm{U}_2]$.

(4)

$$\mathbf{12} \wedge \mathbf{45} \wedge \mathbf{3}\mathrm{U}_2 = [\mathbf{123}][\mathbf{45}\mathrm{U}_2] - [\mathbf{12}\mathrm{U}_2][\mathbf{345}]$$

leads to the final result

$$\begin{aligned} \mathbf{12} \wedge \mathbf{1'2'} \wedge \mathrm{U}_2\mathrm{U}_1\underline{[\mathbf{345}]} \;=\; & \mathbf{1'2'} \wedge \mathbf{45} \wedge \mathrm{U}_2\mathrm{U}_1 \;\underline{[\mathbf{123}]} \\ & + \mathbf{12} \wedge \mathbf{45} \wedge \mathrm{U}_2\mathrm{U}_1 \;\underline{[\mathbf{1'2'3}]} \\ & + \underline{\mathbf{12} \wedge \mathbf{1'2'} \wedge \mathbf{45}}\; [\mathbf{3}\mathrm{U}_2\mathrm{U}_1]. \end{aligned}$$

How to do the same thing in Euclidean geometry?

3. Null Bracket Algebra and Applications

Invariant Computation in Euclidean Geometry

- Algebra of Covariants: Conformal Geometric Algebra.

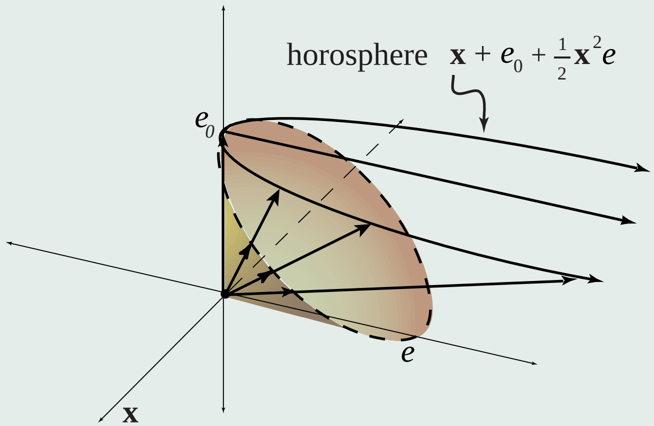
  Composed of null vectors and Minkowski extensors (blades) in the Clifford Algebra of $\mathcal{R}^{n+1,1}$.

  A universal algebra for projective, affine, Euclidean, hyperbolic, elliptic, conformal, geometries.

Classical Conformal Model

Wachter (1840's) – S. Lie (1872).

Non-Euclidean model for Euclidean geometry, non-affine model for affine space:



Let null vector $\mathbf{e}_0$ be the origin, null vector $\mathbf{e}$ be the point at infinity.

A point $\mathbf{x}$ in $\mathcal{R}^n$ is represented by the null vector

$$\mathbf{x}' = \mathbf{e}_0 + \mathbf{x} + \frac{\mathbf{x}^2}{2}\mathbf{e}.$$

The $\mathbf{x}'$s form a set

$$\mathcal{N}_{\mathbf{e}} = \{\mathbf{x} \in \mathcal{R}^{n+1,1} \,|\, \mathbf{x}^2 = 0,\ \mathbf{x} \cdot \mathbf{e} = -1\}.$$

1. $\mathcal{N}_e$ is an isomorphic model: for $\mathbf{x}'_i = \mathbf{e}_0 + \mathbf{x}_i + \frac{\mathbf{x}_i^2}{2}\mathbf{e}$,

$$|\mathbf{x}_i - \mathbf{x}_j| = |\mathbf{x}'_i - \mathbf{x}'_j|.$$

   The inner product is geometrically meaningful:

$$\mathbf{a}' \cdot \mathbf{b}' = -\frac{|\mathbf{a}' - \mathbf{b}'|^2}{2} = -\frac{d_{ab}^2}{2}.$$

   However, this model depends on the choice of the origin.

2. Homogeneous conformal model: $\mathcal{N}$ and $\mathbf{e} \in \mathcal{N}$, where

$$\mathcal{N} = \{\mathbf{x} \in \mathcal{R}^{n+1,1} \,|\, \mathbf{x}^2 = 0\}.$$

   The model is homogeneous and conformal:
   $\mathbf{x} \in \mathcal{N}$ represents a finite point iff $\mathbf{x} \cdot \mathbf{e} \neq 0$.

Conformal Geometric Algebra

The covariant algebra established upon the homogeneous conformal model.

- Representational constituents： Grassmann-Cayley algebra, bracket algebras, Clifford algebra, spinors and twistors.

  Represent: geometric entities, geometric quantities, geometric relations, geometric transforms.

- Computational mechanism: syzygies.
- Feature: covariant representation of geometric entities and transforms;

  invariant hiererchical representation of geometric quantities and relations.

  Keep intrinsic geometric structure in algebraic representation and computation.

- Algebra of Basic Invariants: Inner-Product Bracket Algebra.

  An nD ($n > 1$) inner-product bracket algebra generated by a sequence of $m \geq n$ symbols of vectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$, is

$$\frac{\text{Polynomial ring generated by subsequences of length 2 and } n}{\text{Ideal generated by the following syzygies}}$$

  - GP1: $\sum_{k=1}^{n+1} \mathbf{i} \cdot \mathbf{j}_k [\mathbf{j}_1 \mathbf{j}_2 \cdots \check{\mathbf{j}}_k \cdots \mathbf{j}_{n+1}]$.
  - GP2: $[\mathbf{i}_1 \mathbf{i}_2 \cdots \mathbf{i}_n][\mathbf{j}_1 \mathbf{j}_2 \cdots \mathbf{j}_n] - \det(\mathbf{i}_k \cdot \mathbf{j}_l)_{k,l=1..n}$.

Remark: They are resp. $\mathbf{i} \cdot (\mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_{n+1}) = 0$ and

$$[\mathbf{i}_1 \cdots \mathbf{i}_n][\mathbf{j}_1 \cdots \mathbf{j}_n] = (\mathbf{i}_1 \wedge \cdots \wedge \mathbf{i}_n) \cdot (\mathbf{j}_1 \wedge \cdots \wedge \mathbf{j}_n).$$

- A graded version of Inner-Product Bracket Algebra:

  An nD ($n > 1$) graded inner-product bracket algebra generated by a sequence of $m \geq n$ symbols of vectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$, is

$$\frac{\text{Polynomial ring generated by subsequences of length } n\text{, and} \quad \text{symmetric pairs of subsequences of equal length } i\text{, for } 1 \leq i \leq n}{\text{Ideal generated by the following syzygies}}$$

  - GP1: $\displaystyle\sum_{k=1}^{n+1} (\mathbf{i} \,|\, \mathbf{j}_k)[\mathbf{j}_1\mathbf{j}_2 \cdots \check{\mathbf{j}}_k \cdots \mathbf{j}_{n+1}]$.
  - GP2: $[\mathbf{i}_1\mathbf{i}_2 \cdots \mathbf{i}_n][\mathbf{j}_1\mathbf{j}_2 \cdots \mathbf{j}_n] - (\mathbf{i}_1\mathbf{i}_2 \cdots \mathbf{i}_n \,|\, \mathbf{j}_1\mathbf{j}_2 \cdots \mathbf{j}_n)$.
  - GP3: (Laplace) $(\mathbf{i}_1\mathbf{i}_2 \cdots \mathbf{i}_l \,|\, \mathbf{j}_1\mathbf{j}_2 \cdots \mathbf{j}_l) - \displaystyle\sum_{\{\sigma,\check{\sigma}\}=\{1,2,\ldots,l\}} \mathrm{sign}(\sigma, \check{\sigma})$
    $(\sigma(\mathbf{i})_1\sigma(\mathbf{i})_2 \cdots \sigma(\mathbf{i})_k \,|\, \mathbf{j}_1\mathbf{j}_2 \cdots \mathbf{j}_k)(\check{\sigma}(\mathbf{i})_1\check{\sigma}(\mathbf{i})_2 \cdots \check{\sigma}(\mathbf{i})_{l-k} \,|\, \mathbf{j}_{k+1}\mathbf{j}_{k+2} \cdots \mathbf{j}_l)$.

System of Advanced Invariants

An advanced invariant is a polynomial function of basic invariants.

By putting advanced invariants as independent elements into the system of basic invariants, taking their relations with basic invariants as new syzygies, one obtains a system of advanced invariants.

Purpose: simplify computation and keep geometric meaning.

Advanced invariants should

1. have clear geometric interpretation.
2. be hierarchical.
3. have relatively nice symmetry wrt its constituents.

   This kind of symmetry, if represented by basic invariants, is often a complicated syzygy relation.

Clifford Bracket Algebra:

An nD ($n > 1$) Clifford bracket algebra generated by a sequence of $m \geq n$ symbols of vectors $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{m}$, is

$$\frac{\begin{array}{r}\text{Polynomial ring generated by subsequences of length } n\text{, and} \\ \text{symmetric pairs of vectors, and} \\ \text{repeatable permutations of vectors of length } n+2k \text{ for } k>0\text{, and} \\ \text{another group of repeatable permutations of vectors of length } 2l+2 \text{ for } l>0\end{array}}{\text{Ideal generated by GP1, GP2, SB and AB}}$$

- SB: $[\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{n+2k}] - \sum_{\{\sigma,\check{\sigma}\}=\{1,2,\ldots,n+2k\}} \mathrm{sign}(\sigma,\check{\sigma})\langle\sigma(\mathbf{i})_1\sigma(\mathbf{i})_2\cdots\sigma(\mathbf{i})_{2k}\rangle [\check{\sigma}(\mathbf{i})_1\check{\sigma}(\mathbf{i})_2\cdots\check{\sigma}(\mathbf{i})_n].$
- AB: $\langle\mathbf{i}_1\mathbf{i}_2\cdots\mathbf{i}_{2l}\rangle - \sum_{k=2}^{2l}(-1)^k\langle\mathbf{i}_1\mathbf{i}_k\rangle\,\langle\mathbf{i}_2\cdots\check{\mathbf{i}}_k\cdots\mathbf{i}_{2l}\rangle.$

Null Bracket Algebra

The m symbolic vectors are null: $\langle \mathbf{ii} \rangle = 0$.

Basic invariants in null bracket algebra: 2 kinds.

- Inner product $\langle \mathbf{12} \rangle = \mathbf{1} \cdot \mathbf{2}$ (symmetry wrt permutation).
- Classical bracket $[\mathbf{12} \cdots \mathbf{n}]$ (antisymmetry wrt permutation).

Advanced invariants in null bracket algebra: 3 kinds.

- Angular bracket (also called Pfaffian) $\langle \mathbf{12} \cdots (\mathbf{2k}) \rangle$.
  Shift and reversion symmetry wrt its components.
- Round bracket (also called Gram determinant) $(\mathbf{12} \cdots \mathbf{r} \,|\, \mathbf{1'2'} \cdots \mathbf{r'})$.

$$(\mathbf{12} \cdots \mathbf{r} \,|\, \mathbf{1'2'} \cdots \mathbf{r'}) = \det(\mathbf{i} \cdot \mathbf{j'})_{i,j=1..r}.$$

  Antisymmetry within each group and symmetry between two groups.
- Square bracket $[\mathbf{12} \cdots (\mathbf{n} + \mathbf{2k})]$.
  Antisymmetry wrt shift and reversion.

Geometric Interpretation

The angular and square brackets are cosines and sines of some angels, multiplied by a monomial of distances.

E.g., for distinct points i's in the Euclidean plane,

$$\begin{aligned} \langle \mathbf{1234} \rangle &= -\frac{d_{12}d_{23}d_{34}d_{41}}{2} \cos\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}); \\ [\mathbf{1234}] &= -\frac{d_{12}d_{23}d_{34}d_{41}}{2} \sin\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}). \end{aligned}$$

$$\begin{aligned} \langle \mathbf{123456} \rangle &= -\frac{d_{12}d_{23}d_{34}d_{45}d_{56}d_{61}}{2} \cos(\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}) + \angle(\overrightarrow{\mathbf{145}}, \overrightarrow{\mathbf{156}})); \\ [\mathbf{123456}] &= -\frac{d_{12}d_{23}d_{34}d_{45}d_{56}d_{61}}{2} \sin(\angle(\overrightarrow{\mathbf{123}}, \overrightarrow{\mathbf{134}}) + \angle(\overrightarrow{\mathbf{145}}, \overrightarrow{\mathbf{156}})). \end{aligned}$$

The general case:

$$\langle \mathbf{12\cdots 2l+2}\rangle = -\frac{d_{12}d_{23}\cdots d_{(2l+1)(2l+2)}d_{(2l+2)1}}{2}\cos(\angle(\overrightarrow{\mathbf{123}},\overrightarrow{\mathbf{134}})+\angle(\overrightarrow{\mathbf{145}},\overrightarrow{\mathbf{156}}) + \cdots + \angle(\overrightarrow{\mathbf{1(2l)(2l+1)}},\overrightarrow{\mathbf{1(2l+1)(2l+2)}}));$$

$$[\mathbf{12\cdots 2l+2}] = -\frac{d_{12}d_{23}\cdots d_{(2l+1)(2l+2)}d_{(2l+2)1}}{2}\sin(\angle(\overrightarrow{\mathbf{123}},\overrightarrow{\mathbf{134}})+\angle(\overrightarrow{\mathbf{145}},\overrightarrow{\mathbf{156}}) + \cdots + \angle(\overrightarrow{\mathbf{1(2l)(2l+1)}},\overrightarrow{\mathbf{1(2l+1)(2l+2)}})).$$

$\angle(\overrightarrow{\mathbf{123}},\overrightarrow{\mathbf{134}})$ is the angle at point $\mathbf{1}$ from the tangent direction of oriented circle $\mathbf{123}$ to that of oriented circle $\mathbf{134}$.

Brackets are rational representations of angles and their sums in the plane, in the form of functions of vertices of the angles.

Geometric Representation

Length:

$$\mathbf{a}\cdot\mathbf{b} = -\frac{d_{\mathbf{ab}}^2}{2}.$$

Radius:

$$\rho_{\mathbf{123}}^2 = \frac{[\mathbf{e123}]^2}{(\mathbf{123}|\mathbf{123})}.$$

Area:

$$S_{\mathbf{123}} = \frac{1}{2}[\mathbf{e123}], \quad S_{\mathbf{1234}} = \frac{1}{4}[\mathbf{e13e24}].$$

Angle:

An oriented angle is a 2D rotation, so $\angle\mathbf{123}$ can be represented by three points:

Vertex $\mathbf{2}$, a point $\mathbf{1}$ on initial ray, a point $\mathbf{3}$ on terminal ray.

- An angle is transcendental wrt the coordinates of its three points.
- The sine and cosine of an angle is irrational wrt the three points.
- The tangent of an angle is rational wrt the points.
- An oriented angle modulo π is called a full-angle. A rational description to an angle is accurate up to its full-angle class.

Two full-angles $\angle\mathbf{123}, \angle\mathbf{1'2'3'}$ are equal iff $\tan\angle\mathbf{123} = \tan\angle\mathbf{1'2'3'}$.

In NBA, since

$$[\mathbf{e123}] = 2\, d_{\mathbf{12}} d_{\mathbf{23}} \sin\angle(\mathbf{12}, \mathbf{23}),$$
$$\langle\mathbf{e123}\rangle = 2\, d_{\mathbf{12}} d_{\mathbf{23}} \cos\angle(\mathbf{12}, \mathbf{23}),$$

we have

$$\frac{[\mathbf{e123}]}{\langle\mathbf{e123}\rangle} = \frac{[\mathbf{e1'2'3'}]}{\langle\mathbf{e1'2'3'}\rangle},$$

i.e.,

$$[\mathbf{e123}]\langle\mathbf{e123}\rangle - [\mathbf{e1'2'3'}]\langle\mathbf{e1'2'3'}\rangle = [\mathbf{e123e3'2'1'}] = 0.$$

Longer bracket are indispensible for representing the sum, difference and equality of two angles.

New Techniques

Main idea:

develop a system of advanced geometric invariants, set up the computational mechanism without resorting to low-level invariants.

$$
\begin{array}{rcrcr}
\text{Representation with} & & \text{Computation with} & & \text{Conclusion in the form of} \\
\\
\text{advanced invariants} & \longrightarrow & \text{advanced invariants} & \longrightarrow & \text{advanced invariants} \\
| & & X & & | \\
\downarrow & & X & & \downarrow \\
\text{basic invariants} & \longrightarrow & \text{basic invariants} & \longrightarrow & \text{basic invariants} \\
| & & X & & | \\
\downarrow & & X & & \downarrow \\
\text{coordinates} & \longrightarrow & \text{coordinates} & \longrightarrow & \text{coordinates}
\end{array}
$$

Rational Clifford Expansion

E.g., When $n = 4$ and $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}$ are null vectors,

$$[\mathbf{123456}] = -2\frac{\mathbf{2}\cdot\mathbf{3}[\mathbf{1256}][\mathbf{3456}] + \mathbf{5}\cdot\mathbf{6}[\mathbf{1236}][\mathbf{2345}]}{[\mathbf{2356}]},$$

$$\langle\mathbf{123456}\rangle = -2\frac{\mathbf{2}\cdot\mathbf{3}[\mathbf{1256}]\langle\mathbf{3456}\rangle - \mathbf{5}\cdot\mathbf{6}[\mathbf{1236}]\langle\mathbf{2345}\rangle}{[\mathbf{2356}]}.$$

Rational Clifford Factorization

The inverse of rational Clifford expansion.

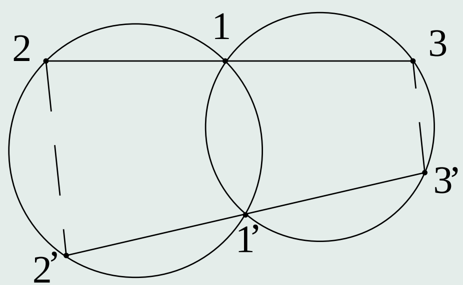
Extremely useful and powerful.

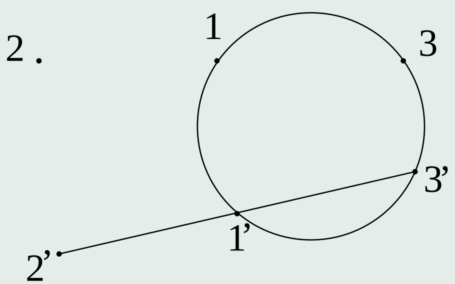
E.g., Miquel's 5-circle factorization:

$$(\mathbf{1}\cdot\mathbf{3})(\mathbf{2}\cdot\mathbf{3})[\mathbf{1245}][\mathbf{1256}]+(\mathbf{1}\cdot\mathbf{5})(\mathbf{2}\cdot\mathbf{5})[\mathbf{1234}][\mathbf{1236}]=-\frac{1}{2}[\mathbf{13241526}][\mathbf{1235}],$$

$$(\mathbf{1}\cdot\mathbf{3})(\mathbf{2}\cdot\mathbf{3})[\mathbf{1245}]\langle\mathbf{1526}\rangle+(\mathbf{1}\cdot\mathbf{5})(\mathbf{2}\cdot\mathbf{5})[\mathbf{1234}]\langle\mathbf{1326}\rangle=\frac{1}{2}\langle\mathbf{13241526}\rangle[\mathbf{1235}].$$

Application: Geometric Factorization and Theorem Completion





Example 1. Let $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{1}', \mathbf{2}'$ be free points in the plane.

Point $\mathbf{3}'$ is the intersection of $\mathbf{1}'\mathbf{2}'$ and $\mathbf{131}'$ other than $\mathbf{1}'$:

$$\mathbf{3}' = \mathbf{1}'\mathbf{13} \cap \mathbf{1}'\mathbf{e2}' = -\mathbf{e}\cdot\mathbf{2}'[\mathbf{e131}'][\mathbf{131}'\mathbf{2}']\mathbf{1}' + \frac{1}{2}[\mathbf{1}'\mathbf{131}'\mathbf{e2}']([\mathbf{131}'\mathbf{2}']\mathbf{e} + [\mathbf{e131}']\mathbf{2}').$$

Substitute into $[\mathbf{e22'e33'}]$ and simplify:

$$\begin{aligned}
&[\mathbf{e22'e33'}] \\
=&-\mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{131'2'}][\mathbf{e22'e31'}]+\frac{1}{2}[\mathbf{1'131'e2'}][\mathbf{e131'}][\mathbf{e22'e32'}] \\
=&\mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}](-[\mathbf{131'2'}][\mathbf{e22'e31'}]+[\mathbf{e232'}][\mathbf{1'131'e2'}]) \\
=&2(\mathbf{e}\cdot\mathbf{2'})[\mathbf{e131'}](\mathbf{e}\cdot\mathbf{3}[\mathbf{e21'2'}][\mathbf{131'2'}]+\mathbf{1'}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{e232'}]) \\
=&\mathbf{e}\cdot\mathbf{2'}[\mathbf{e131'}][\mathbf{e31'2'}][\mathbf{e311'2'2}].
\end{aligned}$$

The discarded conditions $[\mathbf{e123}]$ and $[\mathbf{121'2'}]$ do not appear.

Factor $[\mathbf{e311'2'2}]$ has 6-termed shortest expansion, only two terms containing the missing conditions:

$$\begin{aligned}
r[\mathbf{e311'2'2}] = &\ \mathbf{e}\cdot\mathbf{3}[\mathbf{121'2'}]+\mathbf{e}\cdot\mathbf{1}[\mathbf{231'2'}]+\mathbf{1}\cdot\mathbf{3}[\mathbf{e21'2'}] \\
&+\mathbf{1'}\cdot\mathbf{2'}[\mathbf{e123}]+\mathbf{2}\cdot\mathbf{1'}[\mathbf{e132'}]-\mathbf{2}\cdot\mathbf{2'}[\mathbf{e131'}].
\end{aligned}$$

Rational expansion:

$[\mathbf{e311'2'2}]$ has the following 2-termed rational expansion:

$$\frac{1}{2}[\mathbf{e311'2'2}] = \frac{\mathbf{1}\cdot\mathbf{3}[\mathbf{e232'}][\mathbf{121'2'}] - \mathbf{2}\cdot\mathbf{2'}[\mathbf{e123}][\mathbf{131'2'}]}{[\mathbf{1232'}]}.$$

The missing conditions show up.

Result:

$$\frac{[\mathbf{e22'e33'}]}{\mathbf{e}\cdot\mathbf{3'}} = \frac{[\mathbf{e31'2'}][\mathbf{e311'2'2}]}{\mathbf{1'}\cdot\mathbf{2'}\,[\mathbf{e131'}]}.$$

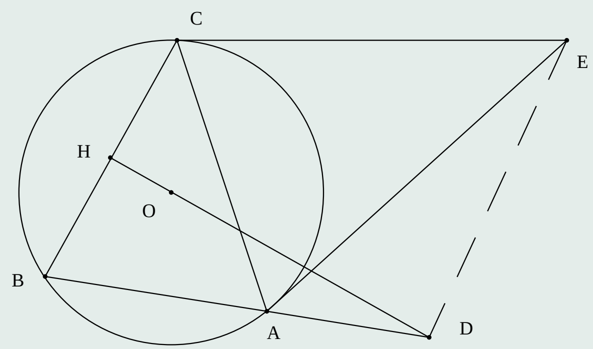
Fail to make it with basic invariants from the 6-termed expansion of $[\mathbf{e311'2'2}]$.

Example 2. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be on circle $\mathbf{O}$,

lines $\mathbf{EA}, \mathbf{EC}$ be tangent to the circle.

$\mathbf{D}$ be intersection of line $\mathbf{AB}$ with perpendicular bisector of $\mathbf{BC}$.

Then $\mathbf{DE} \| \mathbf{BC}$.



Now set $\mathbf{E}, \mathbf{H}$ to be completely free.

Obtain

$$
\begin{aligned}
[\mathbf{eBCeDE}] = \frac{\mathbf{e}\cdot\mathbf{D}}{2\ \mathbf{A}\cdot\mathbf{C}\ [\mathbf{eABeOH}]}\Big(&\mathbf{e}\cdot\mathbf{B}\ \langle\mathbf{ACBAeH}\rangle\ \underline{\langle\mathbf{eECO}\rangle} \\
&+\mathbf{e}\cdot\mathbf{C}\ \langle\mathbf{BACBeH}\rangle\ \underline{\langle\mathbf{eEAO}\rangle} \\
&-2\ \mathbf{A}\cdot\mathbf{B}\ \langle\mathbf{eACB}\rangle\ (\mathbf{eA}|\mathbf{BC})[\mathbf{eACE}]\underline{[\mathbf{eBCH}]} \\
+2\ (\mathbf{A}\cdot\mathbf{B})(\mathbf{e}\cdot\mathbf{C})\langle\mathbf{eACB}\rangle\ [\mathbf{eACE}]\ &\underline{(\mathbf{e}\cdot\mathbf{C}\ [\mathbf{eABH}] + \mathbf{e}\cdot\mathbf{B}\ [\mathbf{eACH}])}\Big).
\end{aligned}
$$

4. Conclusion

- Sturmfels and Whiteley (1991) proved that in bracket algebra, any polynomial when multiplied by a suitable bracket monomial, is Cayley factorizable.

  This is the simplest case of rational factorization.

- Rational expansion is the inverse procedure of rational factorization. It expresses an advanced invariant as a rational polynomial of basic invariants.

  Practically much more powerful. **Highlight of this talk.**

- ...... But many problems are open. Systematic theory is to be set up.

  E.g. Crapo binomial: Let $\mathbf{1}, \mathbf{2}, \cdots \mathbf{k}$ and $\mathbf{1}', \mathbf{2}', \cdots \mathbf{k}'$ be points in the plane. Is

$$f = [\mathbf{1}\mathbf{2}'\mathbf{3}'][\mathbf{2}\mathbf{3}'\mathbf{4}'] \cdots [\mathbf{k}\mathbf{1}'\mathbf{2}'] + (-1)^{k-1}[\mathbf{1}\mathbf{1}'\mathbf{2}'][\mathbf{2}\mathbf{2}'\mathbf{3}'] \cdots [\mathbf{k}\mathbf{k}'\mathbf{1}']$$

  Cayley factorizable? How to find the simplest monomial g s.t. fg is Cayley factorizable?

Thank you for your attention!