On the Mechanization of the Proof of Hessenberg's Theorem

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The stage and the actors

- Axiomatic projective plane geometry
- Pappus: *Axiom*
- Desargues: *Axiom*
- Hessenberg, 1905: Pappus => Desargues, *but the proof was incomplete*
- Cronheim, 1953: *complete proof*

CL as a fragment of FOL

- Coherent formula: $C \Rightarrow D$, where
- $C = A_1 \wedge \ldots \wedge A_n$ ($n \geq 0$, A_i atoms) and
- $D = E_1 \vee \ldots \vee E_m$ ($m \geq 0$), where each
- $E_j = (\sum x_1 \ldots x_k)\ C_j$ ($k \geq 0$ may vary with j, each C_j a conjunction of atoms)
- Implicit universal closure
- Coherent theory = set of coherent formulas

Examples

- Sorted incidence (Horn clauses):
  $x|y \Rightarrow point(x) \wedge line(y)$
- Projective unicity (resolution clause):
  $p|l \wedge p|m \wedge q|l \wedge q|m \Rightarrow p=q \vee l=m$
- Projective meet (coherent clause):
  $point(p) \wedge point(q) \Rightarrow (\sum l)(line(l) \wedge p|l \wedge q|l)$
- In general: $A_1 \wedge \ldots \wedge A_n \Rightarrow$
  $((\sum \mathbf{x})\ A_{11} \wedge \ldots \wedge A_{1i}) \vee \ldots \vee$
  $((\sum \mathbf{y})\ A_{k1} \wedge \ldots \wedge A_{kj})$

Rationale

- Horn clauses: DCG and Prolog
- Resolution: Automated Theorem Proving
- Coherent logic: ATP and ?
  - More expressive than resolution logic
  - Why skolemize, e.g., $p(x,y) \Rightarrow (\sum z)\ p(x,z)$?
  - Constructive logic (Coq!)
  - Natural proof theory/objects

Proof system

- Forward ground reasoning
- Case distinction for ⋁
- Introduction of witnesses for ∑
- Example, prove r when given:
  1. q(x) => false
  2. (∑x) p(x)
  3. p(y) => q(y) ⋁ r

Metaproperties of CL

- Soundness
- Completeness wrt Tarskian semantics
- Constructivity
- Reduction of FOL to CL
- Semidecidability (even without functions!)
- CL with equality is Finite Model Complete
- Automation (SATCHMO!)

Samples of ATP

- Various *.in (input) files enclosed
- To be processed with CL.pl
- Yielding files:
- *.out (output)
- *.prf (intermediate proof format)
- *.v (Coq proof object)

Pappus in CL

```
%Pappus: if A,B,C on L, D,E,F on M, then ...
  i(A,L),i(B,L),i(C,L),i(D,M),i(E,M),i(F,M),
  % cross BF,CE in G
  i(B,N),i(F,N),i(G,N),i(C,O),i(E,O),i(G,O),
  % cross BD,AE in H
  i(B,P),i(D,P),i(H,P),i(A,Q),i(E,Q),i(H,Q),
  % cross CD,AF in I
  i(C,R),i(D,R),i(I,R),i(A,S),i(F,S),i(I,S)
=> % degenerations
  l(N,O);l(P,Q);l(R,S);
  % or G,H,I on some line T
  (∑T) l(T,T),i(G,T),i(H,T),i(I,T).
```

Desargues

- Page 6 in paper

Hessenberg

- Page 13 in paper