CLASSICAL VERSUS COMPUTER METHODS OF SOLVING PROBLEMS IN GEOMETRY

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Introduction

Objectives of the text are essentially two. One of them is to give basics of the theory of automatic theorem proving and apply this theory to examples. The second objective is to show the interesting topics of elementary geometry which the author met in the course of years.

The content consists of six mutually independent stories, which are aimed at a certain theme. Each of these stories was thoroughly studied by the author in the past. Now an attempt was made all these themes to put together.

Each problem is firstly solved by the method of automatic theorem proving, in the second step the problem is solved classically - without using computer (of course, if it is possible).

The author's aim was to acquaint the reader with new computer methods of proving, deriving and discovery of theorems as well as to make him/her think about the inner beauty of classical solutions and about some new knowledge.

In the seminar which I have led in the last years at the University of South Bohemia we solved on computers a number of problems of elementary geometry by the theory of automatic theorem proving. Students were mostly at their 4th year of study, i.e., they had basic knowledge of geometry.

Both by computers and in a classical way we investigated a number of tasks - the formula of Heron for the area of a triangle and its generalization - the formula of Brahmagupta for the area of an inscribed quadrilateral in terms of its lengths of sides, the formula of Staudt, Simson - Wallace theorem and its generalization, Napoleon's theorem and further similar problems.

The first story "Generalization of the formula of Heron" deals with a generalization of Heron's and Brahmagupta's formulas for the area of a triangle and an inscribed quadrilateral in terms of the lengths of sides on an inscribed pentagon.

This problem which was solved in 1994 by Robbins, see D.P. Robbins: *Areas of polygons inscribed in a circle.* Discrete Comput. Geom. **12** (1994), 223-236, is investigated in this book, unlike Robbins, by computer method.

In the second story "Simson - Wallace Theorem" this well known theorem of planar geometry is generalized into three dimensional space.

First known generalizations of the Simson - Wallace theorem in a plane, from which particularly the Guzmán's generalization deserves attention, are demonstrated.

Furthermore two spatial analogies of this theorem, which lead to the cubic surfaces with very interesting properties, are shown.
Various generalizations of Ceva's, Menelaus' and Euler's theorems in a plane and space are studied in the story "Transversals in a polygon". Both a power of a computer approach, which consists in searching for new formulas and a power of a traditional and possible a little forgotten the area method, are shown.

The Napoleon's theorem and its generalization - Petr's theorem - and their planar and spatial analogies are given in the section "Petr - Douglas - Neumann's theorem".

As the name says, this theorem is connected with the well known Czech mathematician, a professor of the Charles University in Prague K. Petr, who first published the theorem in 1905.

In the story "Geometric inequalities" the inequality between lengths of sides and diagonals of a polygon in a plane and space is gradually generalized.

The base is the well known parallelogram law, which played an important role in the thirties of the last century when there was shown that Banach space, in which parallelogram law holds, is the Hilbert space.

In this chapter also the well - known Euler's inequality between the radii of the circumcircle and the incircle of a triangle is explored.

The next story "Regular polygons" is devoted to problems connected with regular polygons. Although it could seem that all important things about regular polygons have been said, it is not this case.

In 1969 two chemists visited the well known mathematician Van der Waerden and stated that according to their investigations an equilateral and equiangular pentagon in a space must necessary be planar. During a short period Van der Waerden and a number of further mathematicians proved that a regular polygon with an odd number of vertices has always an even dimension. In our story this problem is solved by the theory of automatic theorem proving and discovering.

In the last chapter "Miscellaneous" four problems are solved. One of them is a construction which is difficult to solve by classical means as a ruler and compass. By means of computer we are able to solve even such problems which have been taboo in the past, since they have been considered as (Euclidean) unsolvable. This topic would obviously deserve a special tract.

With all problems also classical solutions are mostly given. If a classical solution is missing it is likely caused by the fact, that the author does not know it.

Examples are drawn in dynamic software Cabri II plus. All computations were mostly done using software CoCoA on a computer Intel Pentium 2.00GHz/1572MB RAM.

Generalization of the formula of Heron

Every student knows the formula of Heron for the area p of a triangle with sides of lengths a, b, c

$$p = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

where $s = 1/2(a+b+c)$. This formula is usually ascribed to Heron of Alexandria c. 60 B.C., although it was likely known to Archimedes, 287 - 212 B.C.

The formula of Brahmagupta (Brahmagupta - Indian mathematician, 598 - c. 665 A.D.) for the area p of a convex cyclic quadrilateral which is given by the lengths of sides a, b, c, d

$$p = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{2}$$

where $s = 1/2(a+b+c+d)$, is a generalization of the formula of Heron.

Since that time, despite a great effort of mathematician from all over the world, no similar formula for the area of a cyclic n- gon for $n > 4$, has appeared. Until 1994 when American D.P.Robbins published [82]. Almost 1400 years the formula for the area of a cyclic pentagon was missing.

In the meantime some works on computation of the area of cyclic polygons of special classes appeared, see for example [5], [16], [86]. Main reason why such a long time elapsed is a big complexity of such formulas.

Area of a cyclic pentagon

Denote $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DE|$, $e = |EA|$, $p =$ area of a pentagon Fig. 1. We get a polynomial of 14th degree in p, which contains 6672 terms. A substitution of the elementary symmetric functions

$k = \sum a^2, l = \sum a^2b^2, m = \sum a^2b^2c^2, n = \sum a^2b^2c^2d^2, o = a^2b^2c^2d^2e^2,$

where $q = 16p^2$, leads to the equation $h = 0$ which contains 153 terms and is still too long to write it completely. It begins with

$h : q^7 + q^6(7k^2 - 24l) + q^5(21k^4 - 144k^2l + 240l^2 + 16km - 192n) + \ldots = 0.$

If we denote $K = k^2 - 4l + q, L = kK + 8m, M = K^2 - 64n, N = 128o$

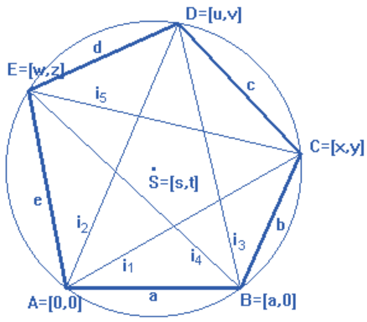

Figure 1: The area of a cyclic pentagon - convex case

then elimination of variables k, l, m, n, o in the ideal (h, K, L, M, N) leads to the formula which contains only 5 terms. We arrived at the following theorem (Robbins):

Given a cyclic pentagon with the lengths of sides a, b, c, d, e and the area p. Let $q = 16p^2$ and L, M, N be as above. Then q obeys the equation

$$L^2M^2 + M^3q - 16L^3N - 18LMNq - 27N^2q^2 = 0 \ . \qquad (3)$$

The relation (3) can be considered as a generalization of the formulas of Heron and Brahmagupta.

Radius of a cyclic pentagon

We will compute the circumradius r of a cyclic pentagon $ABCDE$ with the given sides a, b, c, d, e and let $s = r^2$.

In 17 seconds in CoCoA we receive (4). We can state the following theorem:

A pentagon with the sides a, b, c, d, e which is inscribed in the circle with the radius r is given. Let s, k, l, m, n, o are as above. Then

$$s^3[(s(k^2 - 4l) + m)^2 - n(8s - k)^2]^2 + os^2Q + o^2sP + o^3 = 0 \qquad (4)$$

holds, where P, Q are polynomials in k, l, m, n.

Remarks

1. Two given algorithms Cyclic Pentagon Area Algorithm and Cyclic Pentagon Radius Algorithm could serve as a tool for computing of the area and radius of a cyclic n-gons for $n \geq 6$.

2. By using this method two main problems occurred: a) Still a big CPU time spent for computations. b) Finding an appropriate expressions like K, L, M, N to abbreviate the final polynomial.

Simson - Wallace theorem

There is a nice property of the circumcircle of a triangle, which is often ascribed to Scottish mathematician R. Simson (1687–1768), but it was really discovered by another Scottish mathematician W. Wallace (1768–1843) in 1799, see [16].

Let ABC be a triangle and P a point of the circumcircle of ABC. Then the feet of perpendiculars from P onto the sides of ABC lie on a straight line, see Fig. 2.

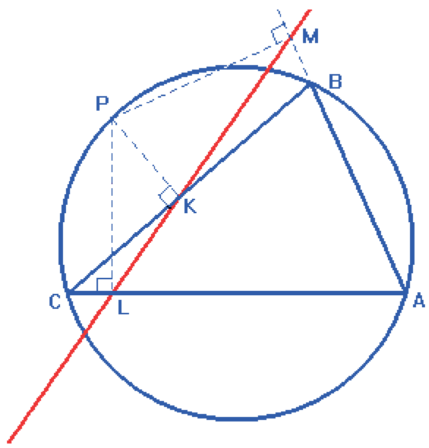

Figure 2: Points K, L, M are collinear

Generalization of Guzmán

The theorem which is due to M. de Guzmán [33] reads :

Project P onto the sides BC, AC, AB of a triangle ABC in the given directions u, v, w which are not parallel to the sides BC, AC, AB into the points K, L, M respectively. We also exclude the case $u \parallel v \parallel w$. Then the locus of points P such that the area of a triangle K, L, M equals s is the conic $C(s)$.

Let us choose the Cartesian system of coordinates so that, Fig. 3: $A = [a, 0]$, $B = [b, c]$, $C = [0, 0]$, $P = [p, q]$, $K = [k_1, k_2]$, $L = [l_1, l_2]$, $M = [m_1, m_2]$, $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, $\mathbf{w} = (w_1, w_2)$.
From $K = P + t_1\mathbf{u}$, $L = P + t_2\mathbf{v}$, $M = P + t_3\mathbf{w}$, $K = C + s_1(B - C)$, $L = C + s_2(A - C)$, $M = A + s_3(B - A)$, where $t_1, t_2, t_3, s_1, s_2, s_3$ are

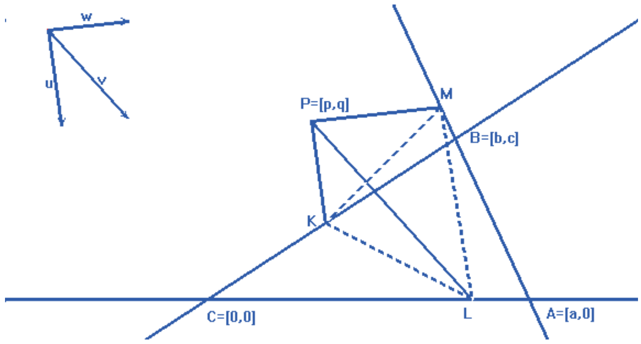

Figure 3: Guzmán's generalization of Simson - Wallace theorem

real parameters, we get the system of equations:

$h_1: k_1 = p + t_1u_1,\ h_2: k_2 = q + t_1u_2,\ h_3: l_1 = p + t_2v_1,\ h_4: l_2 = q+t_2v_2,\ h_5: m_1 = p+t_3w_1,\ h_6: m_2 = q+t_3w_2,\ h_7: k_1 = s_1b,\ h_8: k_2 = s_1c,\ h_9: l_1 = s_2a,\ h_{10}: l_2 = 0,\ h_{11}: m_1 = a+s_3(b-a),\ h_{12}: m_2 = s_3c.$

The conclusion h_{13} is given by

area of $KLM = s \Leftrightarrow h_{13}: 2s = k_1l_2+l_1m_2+m_1k_2-m_1l_2-k_1m_2-l_1k_2.$

We enter

```
UseR::=Q[abcpqfu[1..2]v[1..2]w[1..2]k[1..2]l[1..2]m[1..2]
t[1..3]s[1..3]];
I:=Ideal(k[1]-p-t[1]u[1],k[2]-q-t[1]u[2],l[1]-p-t[2]v[1],l[2]-q
-t[2]v[2],m[1]-p-t[3]w[1],m[2]-q-t[3]w[2],k[1]-s[1]b,k[2]-s[1]c,
l[1]-s[2]a,l[2],m[1]-a-s[3](b-a),m[2]-s[3]c,k[1]l[2]+l[1]m[2]+
m[1]k[2]-m[1]l[2]-k[1]m[2]-l[1]k[2]-2s);
Elim(k[1]..s[3],I);
```

and get the only algebraic equation of the second degree in p, q

$$C(s) = 0, \tag{5}$$

where

$C(s) = c^2v_2p^2(u_1w_2-u_2w_1)+cpq[au_2(v_1w_2-v_2w_1)-(cv_1+bv_2)(u_1w_2-u_2w_1)] + cq^2[(bv_1(u_1w_2 - u_2w_1) - au_1(v_1w_2 - v_2w_1)] - ac^2v_2p(u_1w_2 - u_2w_1)+acq[cu_1(v_1w_2-v_2w_1)+bw_2(u_1v_2-u_2v_1)]+2v_2s(cu_1-bu_2)(cw_1+w_2(a-b)).$

In Fig. 4 a hyperbola for directions u, v, w and $s = 0$ is depicted.

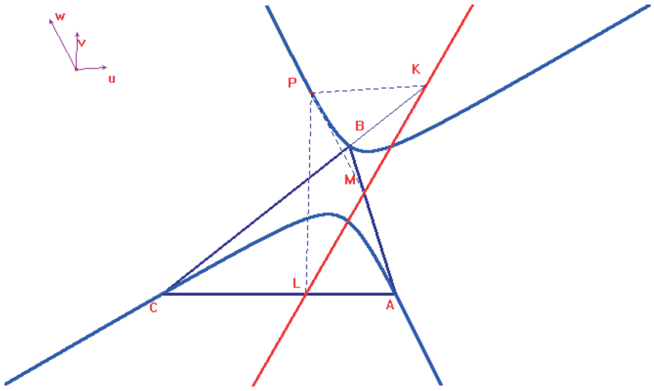

Figure 4: The curve $C(u, v, w, 0)$ is a hyperbola

Further on we see an ellipse for directions u, v, w and $s \neq 0$, Fig. 5.

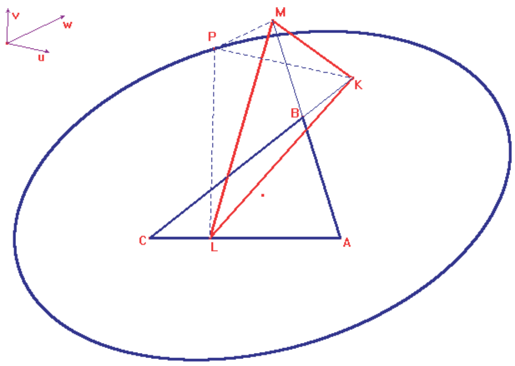

Figure 5: For directions u, v, w and $s \neq 0$ we get an ellipse

Generalization on a tetrahedron

Consider a tetrahedron $ABCD$ in the Euclidean space E^3. Let P be an arbitrary point and K, L, M, N the feet of perpendiculars dropped from P onto the faces BCD, ACD, ABD, ABC of the tetrahedron $ABCD$ respectively. We are looking for a locus of points P such that the volume of the tetrahedron $KLMN$ equals to the constant s, cf. Lozano [81].

Using successive elimination we obtain the only condition

$$F(s) = ac^2 f^3 G + s \cdot Q, \qquad (6)$$

where

$G = bf^2q^3(b-a) + fr^3(abe - acd + cd^2 - b^2e - c^2e + ce^2) + c^2f^2p^2q +$

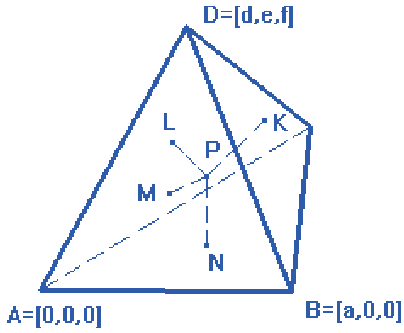

Figure 6: Generalization of Simson - Wallace theorem on a tetrahedron

$cfp^2r(e^2-ce+f^2)+cf^2q^2p(a-2b)+fq^2r(abe-acd+cd^2-b^2e+cf^2)+cf^2r^2p(a-2d)+f^2r^2q(b^2-ab+c^2-2ce)+2cefpqr(b-d)+abcf^2q^2+r^2(abce^2-ac^2de+c^2d^2e+acde^2-2bcde^2-abe^3+b^2e^3+acdf^2-abef^2+b^2ef^2+c^2ef^2)-ac^2f^2pq+acfpr(ce-e^2-f^2)+fqr(ac^2d-2abce-c^2d^2+2bcde-b^2e^2+abe^2+abf^2-b^2f^2-c^2f^2)$

and

$Q=-6(e^2+f^2)((cd-be)^2+b^2f^2+c^2f^2)(a^2c^2-2ac^2d+c^2d^2-2a^2ce+2abce+2acde-2bcde+a^2e^2-2abe^2+b^2e^2+a^2f^2-2abf^2+b^2f^2+c^2f^2),$

which is a constant which doesn't depend on p, q, r, s.

We have the following theorem:

Let K, L, M, N be othogonal projections of an arbitrary point P consecutively on the faces BCD, ACD, ABD, ABC of a tetrahedron $ABCD$. Then the points P such that the tetrahedron $KLMN$ has constant volume s belong to the surface $F(s)=0$.

Now we will give some properties of a surface $F(s)=0$ for s being zero, i.e., when K, L, M, N are complanar. From (6) $F(0)=0 \Leftrightarrow G=0$ follows.

The surface G has following properties [70]:

a) G contains the edges AB, AC, AD, BC, BD, CD of $ABCD$, i.e., G is a circumsurface of $ABCD$.

b) G is a cubic surface.

c) G has 4 singular points - the vertices A, B, C, D of the tetrahedron.

d) The point which is in the intersection of three planes which pass *through the edges AB, BD, DA and are perpendicular to the planes ABC, BDC, DAC belongs to the surface G. Similarly we will proceed for another triple of edges.*

e) The lines AB, AC, AD, BC, BD, CD are torsal lines of the cubic G. The tangent planes at three pairs of opposite edges intersect at another 3 straight lines which are complanar. Each of these three lines intersects the pair of skew torsal lines.

f) There exists a simple rational parametrization of G.

For special values $a = 1, b = 0, c = 1, d = 0, e = 0, f = 1, s = 0$ we get from (6)

$$p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr = 0. \qquad (7)$$

This surface can be easily rationally parametrized taking into account

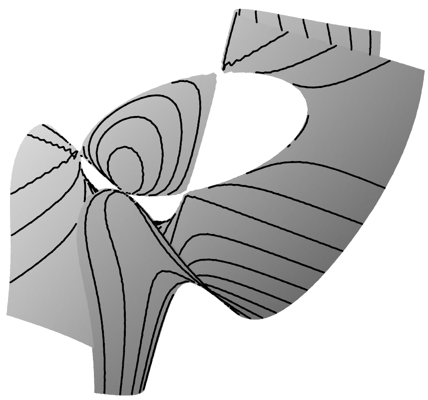

Figure 7: Cubic surface $p^2q + pq^2 + p^2r + q^2r + pr^2 + qr^2 - pq - pr - qr = 0$ as the locus of points P with complanar feet of a (special) tetrahedron

that the surface has 4 double points. Putting $p = ur, q = vr, r = r$ and setting this into (7) we get

$$\begin{aligned} p &= \frac{u(u + uv + v)}{u^2v + u^2 + uv^2 + v^2 + u + v} \\ q &= \frac{v(u + uv + v)}{u^2v + u^2 + uv^2 + v^2 + u + v} \\ r &= \frac{u + uv + v}{u^2v + u^2 + uv^2 + v^2 + u + v} \end{aligned} \qquad (8)$$

for real parameters u, v.

The choice $a = 2, b = 1, c = \sqrt{3}, d = 1, e = 1/\sqrt{3}, f = \sqrt{8/3}$ with the centroid of $ABCD$ in the origin gives for an arbitrary s a one-parametric system of surfaces which are associated with a regular tetrahedron (writing x, y, z instead of p, q, r):

$$24\sqrt{6}x^2y + 24\sqrt{3}x^2z + 24\sqrt{3}y^2z - 8\sqrt{6}y^3 - 16\sqrt{3}z^3 + \\ +36\sqrt{2}x^2 + 36\sqrt{2}y^2 + 36\sqrt{2}z^2 - 18\sqrt{2} - 729s = 0. \tag{9}$$

In Fig. 8 we see a cubic surface (9) associated with a regular tetrahedron $ABCD$ for $s = 10\sqrt{2}/729$.

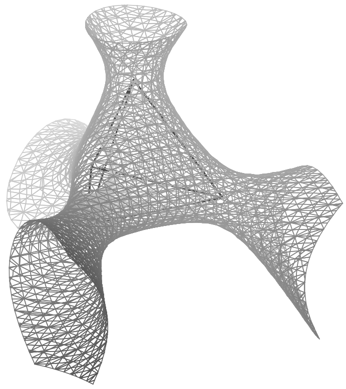

Figure 8: Cubic surface associated with a regular tetrahedron as the locus of points P with constant volume $s = 10\sqrt{2}/729$ of a tetrahedron $KLMN$.

Generalization on a skew quadrilateral

Let P be an arbitrary point and K, L, M, N the feet of perpendiculars dropped from P onto the sides AB, BC, CD, DA of a skew quadrilateral $ABCD$ respectively. Then a point $P = [p, q, r]$ such that K, L, M, N are complanar obeys the equation $H = 0$.

The successive elimination gives a cubic surface $H(p, q, r)$ which contains 176 terms.

For the choice $a = 1, b = 0, c = 1, d = 0, e = 0, f = 1$ we get the cubic surface

$$-p^2q + pq^2 - p^2r - q^2r + pr^2 + qr^2 + p^2 - r^2 - p + r = 0, \tag{10}$$

or after factorization

$$(p - r)(pq - q^2 + pr + qr - p - r + 1) = 0.$$

Thus the cubic surface decomposes into the plane and one sheet hyperboloid.

Remarks:

1) The generalization above stimulates immediately the following question: The 6 edges of a tetrahedron allow three possibilities of skew edge quadrilaterals. How are the three solution surfaces, to these three possibilities related?

2) For the construction of the generalization above it is not essential that the four edges form a skew quadrilateral. One could equally treat the case of four skew given lines a, b, c, d and ask for complanar pedal points K, L, M, N of a point P. And now it would be interesting to know if it makes a difference whether the given lines are generators of a regulus or not.

3) In the last example the respective cubic surface decomposes into a quadric and a plane. In another cases for instance for $a = 1, b = 1, c = 1, d = 0, e = 0, f = 1$ we get an irreducible cubic. Why? □

Transversals in a polygon

Theorem of Ceva:

Given a triangle ABC and three points A', B', C' on the lines BC, AC, AB. Then the lines AA', BB', CC' intersect at one point (which may be in infinity) if and only if

$$(ABC') \cdot (BCA') \cdot (CAB') = -1. \qquad (11)$$

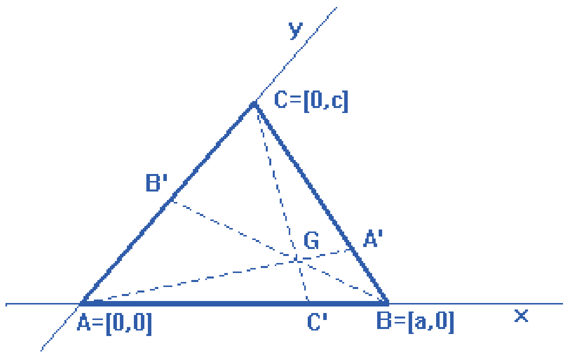

Figure 9: Straight lines AA', BB', CC' are concurrent

Generalization of Ceva's theorem

Let $ABCDE$ be a planar pentagon and M an arbitrary point in the plane of a pentagon. Denote M_1, M_2, M_3, M_4, M_5 intersections of the lines AM, BM, CM, DM, EM with the sides CD, DE, EA, AB, BC or their extensions respectively. What is the relation between ratios of points (CDM_1), (DEM_2), (EAM_3), (ABM_4), (BCM_5)?

By succesive elimination we get get the following generalization of Ceva's theorem [31]:

Let $ABCDE$ be a planar pentagon and M be an arbitrary point of the plane of a pentagon. Let M_1, M_2, M_3, M_4, M_5 be intersection of lines AM, BM, CM, DM, EM with sides CD, DE, EA, AB, BC or their extensions respectively. Then

$$(ABM_4) \cdot (BCM_5) \cdot (CDM_1) \cdot (DEM_2) \cdot (EAM_3) = -1. \qquad (12)$$

Menelaus' theorem

The theorem of Menelaus (Menelaus, 1st century A.D.) is a well - known theorem of elementary geometry in the plane. It reads:

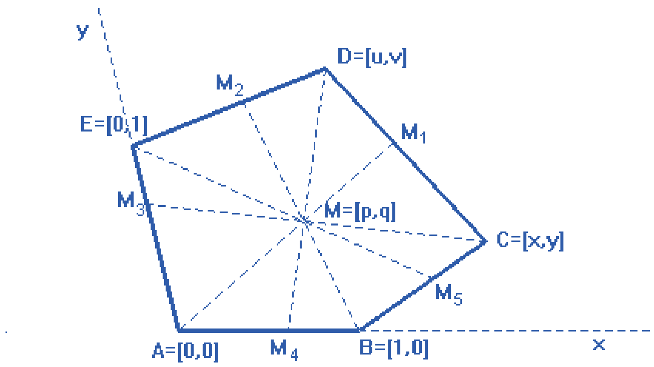

Figure 10: Ceva's theorem for a pentagon

Let A', B', C' be three points on sides BC, CA, AB or their extensions of a triangle ABC respectively. Then the points A', B', C' are collinear if and only if

$$(ABC') \cdot (BCA') \cdot (CAB') = 1. \tag{13}$$

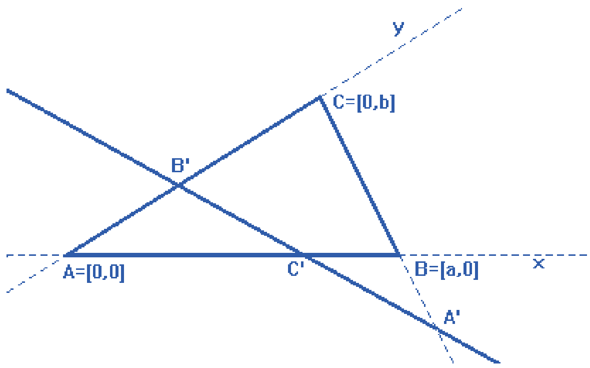

Figure 11: Menelaus' theorem

Generalization of Menelaus' theorem in space

Given a skew quadrilateral $ABCD$ and on its sides AB, BC, CD, DA or their extensions points K, L, M, N respectively. Find a necessary and sufficient condition for the ratios of points $(ABK), (BCL), (CDM), (DAN)$ such that points K, L, M, N to be complanar.

We proved the theorem:

Theorem: *Given a skew quadrilateral $ABCD$ and on its sides AB, BC, CD, DA or their extensions points K, L, M, N respectively. The points K, L, M, N are complanar if and only if*

$$(ABK) \cdot (BCL) \cdot (CDM) \cdot (DAN) = 1. \tag{14}$$

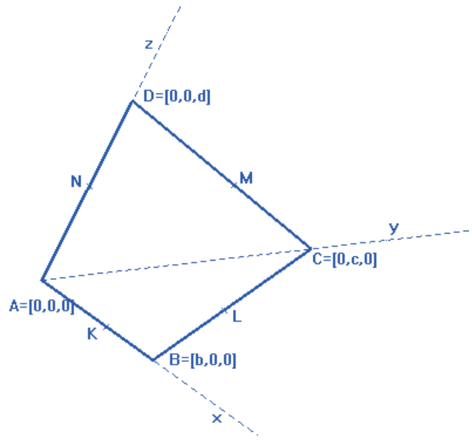

Figure 12: Menelaus' theorem in space

Theorem of Euler

The following theorem is a generalization of the well-known property of medians in a triangle. If AA', BB', CC' are the medians in a triangle ABC then $|GA'|/|AA'| = |GB'|/|BB'| = |GC'|/|CC'| = 1/3$, hence it holds the equality

$$\frac{|GA'|}{|AA'|} + \frac{|GB'|}{|BB'|} + \frac{|GC'|}{|CC'|} = 1, \tag{15}$$

where G is the centroid of a triangle ABC. It is interesting that this property holds not only for the centroid but for an arbitrary *inner* point G of a triangle. In addition if we use in (15) the signed lengths then the theorem is valid even for *all* points of the plane of a triangle, for which (15) is defined.

Spatial analog of Euler's theorem

Now we will show the analog of Euler's relation in the space.

Assume that a tetrahedron $ABCD$ and an arbitrary point O are given. Let A', B', C', D' be intersections of lines AO, BO, CO, DO with planes BCD, ACD, ABD, ABC respectively. Denote

$$k_1 = \frac{\|AO\|}{\|OA'\|}, k_2 = \frac{\|BO\|}{\|OB'\|}, k_3 = \frac{\|CO\|}{\|OC'\|}, k_4 = \frac{\|DO\|}{\|OD'\|}. \tag{16}$$

Find relation which holds for ratios k_1, k_2, k_3, k_4.

By successive elimination we get the relation

$$k_1k_2k_3k_4 = (k_1k_2+k_1k_3+k_2k_3+k_1k_4+k_2k_4+k_3k_4)+2(k_1+k_2+k_3+k_4)+3. \tag{17}$$

We discovered the theorem:

Theorem: *Given a tetrahedron $ABCD$ and an arbitrary point O. Let A', B', C', D' be intersections of lines AO, BO, CO, DO with planes BCD, ACD, ABD, ABC respectively. Denote*

$$k_1 = \frac{\|AO\|}{\|OA'\|}, k_2 = \frac{\|BO\|}{\|OB'\|}, k_3 = \frac{\|CO\|}{\|OC'\|}, k_4 = \frac{\|DO\|}{\|OD'\|}. \tag{18}$$

Then for k_1, k_2, k_3, k_4 the equality (17) holds.

Petr - Douglas - Neumann's theorem

Petr - Douglas - Neumann's theorem (PDN theorem) has a rich history [55], [56], [66], [68], [71], [28]. Its name is closely connected with the name of the Czech mathematician Karel Petr, who first published this theorem in 1905, see [74]. Perhaps, because of the fact that the work [74] was written in Czech (although two years later a German version appeared), for a long time as the authors of this theorem J. Douglas [18] and B.H. Neumann [62], who published their works in the 40th of the last century, were supposed. PDN theorem as the Petr's theorem is also involved in [58]. The name Petr - Douglas - Neumann's theorem is given by H. Martini in 1996 in his paper [56].

Napoleon's theorem

This theorem is often ascribed to the well-known emperor Napoleon Bonaparte [24], [87], although there are doubts about his knowing enough geometry to prove this theorem.
Napoleon's theorem reads:

Over the sides of an arbitrary triangle construct equilateral triangles (all outwardly or inwardly). Then the centers of these equilateral triangles form an equilateral triangle.

Choose the Cartesian coordinate system so that $A = [0, 0]$, $B = [a, 0]$,

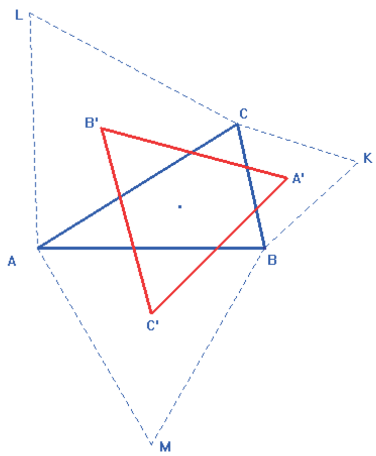

Figure 13: Napoleon's theorem - the triangle $A'B'C'$ is equilateral

$C = [b, c]$, $A' = [k_1, k_2]$, $B' = [l_1, l_2]$, $C' = [m_1, m_2]$ and construct over sides of ABC e.g. outwardly *arbitrary* similar isosceles triangles

ABC', BCA', CAB'. The problem of this task consists in the expression of the notion "outwardly" only by algebraic *equations.* In this case we will apply the method which is due to D. Wang [101].

The vertex A' is the endpoint of a vector whose initial point is in the center of BC with the length $v|BC|$, where v is an arbitrary number, and the same direction as the vector $\overrightarrow{CB}$ rotated by the angle $90°$ in a positive sense. Whence for coordinates k_1, k_2 of the point A' the relation $(k_1 - (a+b)/2, k_2 - c/2) = v(c, a-b)$ holds. Analogous relations hold for other vertices. The number v is the same on all three sides because of similarity of triangles ABC', BCA', CAB'. For the coordinates of vertices A', B', C' we get the following conditions:

$h_1 : 2k_1 - a - b - 2vc = 0,$
$h_2 : 2k_2 - c - 2va + 2vb = 0,$
$h_3 : 2l_1 - b + 2vc = 0,$
$h_4 : 2l_2 - c - 2vb = 0,$
$h_5 : 2m_1 - a = 0,$
$h_6 : m_2 + va = 0.$

We are looking for such a real number v for which the triangle $A'B'C'$ becomes equilateral. As a conclusion we require $|A'B'| = |A'C'|$ and $|A'B'| = |B'C'|$, i.e.,

$h_7 : (k_1 - l_1)^2 + (k_2 - l_2)^2 - (m_1 - k_1)^2 + (m_2 - k_2)^2 = 0,$
$h_8 : (k_1 - l_1)^2 + (k_2 - l_2)^2 - (m_1 - l_1)^2 + (m_2 - l_2)^2 = 0.$

First we require the validity of the equality $|A'B'| = |A'C'|$. In the ideal $I = (h_1, h_2, \ldots, h_7)$ we eliminate dependent variables $k_1, k_2, l_1, l_2, m_1, m_2$

```
Use R::=Q[k[1..2]l[1..2]m[1..2]abcv];
I:=Ideal(2k[1]-a-b-2vc,2k[2]-c-2va+2vb,2l[1]-b+2vc,2l[2]-c-2vb,
2m[1]-a,m[2]+va,(k[1]-l[1])^2+(k[2]-l[2])^2-(m[1]-k[1])^2-(m[2]
-k[2])^2);
Elim(k[1]..m[2],I);
```

and the resulting elimination ideal gives the only condition

$$(12v^2 - 1)(a^2 - b^2 - c^2) = 0,$$

from which we get:

a) $a^2 = b^2 + c^2$, i.e., $|AB| = |AC|$, hence the triangle ABC is isosceles, and v is an *arbitrary* number
or
b) $v = \sqrt{3}/6$ or $v = -\sqrt{3}/6$ and the triangle ABC is an arbitrary.

If a triangle ABC is isosceles then the theorem obviously holds for every v. However we are more interested in the values $v = \sqrt{3}/6$ and $v = -\sqrt{3}/6$ which means that for the values $v = \sqrt{3}/6$ or $v = -\sqrt{3}/6$ the equality $|A'B'| = |A'C'|$ holds.

For $v = \sqrt{3}/6$ we acquire the *outer Napoleon's triangle* $A'B'C'$, for $v = -\sqrt{3}/6$ we get the *inner Napoleon's triangle* $A''B''C''$. The previ-

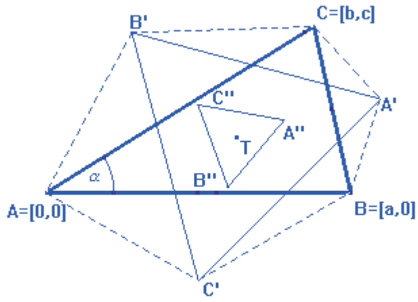

Figure 14: Outer and inner Napoleon's triangles

ous theorem has a wide application. For various values of the angle φ we get various similar isosceles triangles, which correspond to various points S.

PDN theorem for n-gons reads:

Let $j_1, j_2, \ldots, j_{n-1}$ *be any order of numbers* $1, 2, \ldots, n-1$. *Over the sides of an arbitrary plane* n*-gon* P *construct isosceles triangles with vertex angle* $j_1 \cdot 2\pi/n$. *Resulting vertices form a new polygon* P_{j_1}. *Over the sides of* P_{j_1} *construct isosceles triangles with vertex angle* $j_2 \cdot 2\pi/n$. *We get a polygon* P_{j_1,j_2}. *Continuing in this construction for all values* $j_1, j_2, \ldots, j_{n-1}$, *then the final polygon* $P_{j_1,j_2,\ldots,j_{n-1}}$ *is a point - the common centroid of all polygons* P_{j_1}, P_{j_1,j_2}, $\ldots$, $P_{j_1,j_2,\ldots,j_{n-1}}$, *and the* n*-gon* $P_{j_1,j_2,\ldots,j_{n-2}}$ *is* j_{n-1} *- regular.*

PDN theorem in space

We will generalize PDN theorem to n-gons in the 3D space. A spatial generalization of PDN theorem will be shown on a skew pentagon.

Douglas' pentagon

In 1960 J. Douglas published a theorem [18] which is a spatial generalization of PDN theorem. By this theorem we are able, roughly spoken, to assign to an arbitrary skew pentagon a planar affine regular pentagon.

Douglas' theorem reads:

Given a skew pentagon $ABCDE$. Denote by M_1, M_2, M_3, M_4, M_5 the midpoints of sides which are opposite to the vertices A, B, C, D, E respectively. On the ray AM_1 outwardly construct a point A_1 such that $|M_1A_1| = 1/\sqrt{5}|AM_1|$. On BM_2 outwardly construct a point B_1 such that $|M_2B_1| = 1/\sqrt{5}|BM_2|$, etc. until we obtain a pentagon $A_1B_1C_1D_1E_1$. Similarly, on the ray AM_1 inwardly construct a point A_1' such that $|M_1A_1'| = 1/\sqrt{5}|AM_1|$, etc., until we obtain a pentagon $A_1'B_1'C_1'D_1'E_1'$. Then both of the pentagons $A_1B_1C_1D_1E_1$ and $A_1'B_1'C_1'D_1'E_1'$ are planar affine regular.

We will "discover" the theorem by computer, Fig. 15. Since it deals

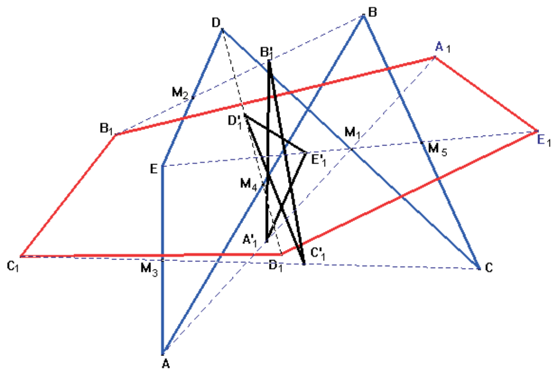

Figure 15: Douglas' pentagon $A_1B_1C_1D_1E_1$

with a problem of affine geometry in space (here only ratios of distances occur) we can choose the affine coordinate system such that $A = [0, 0, 0], B = [x, 0, 0], C = [0, y, 0], D = [0, 0, z], E = [u, v, w]$.
For the midpoints we have $M_1 = [0, y/2, z/2]$, $M_2 = [u/2, v/2, (z + w)/2]$, $M_3 = [u/2, v/2, w/2]$, $M_4 = [x/2, 0, 0]$, $M_5 = [x/2, y/2, 0]$. On the ray AM_1 outwardly we construct a point A_1 such that $|M_1A_1| = k|AM_1|$, where k is an unknown real number. In this way we acquire the points A_1, B_1, C_1, D_1, E_1. Denoting $A_1 = [a_1, a_2, a_3]$, $B_1 = [b_1, b_2, b_3]$, $C = [c_1, c_2, c_3]$, $D_1 = [d_1, d_2, d_3]$, $E_1 = [e_1, e_2, e_3]$ then

$A_1 - M_1 = k(M_1 - A)$, $B_1 - M_2 = k(M_2 - B)$, $C_1 - M_3 = k(M_3 - C)$, $D_1 - M_4 = k(M_4 - D)$, $E_1 - M_5 = k(M_5 - E)$.

We are looking for such a real k that the points A_1, B_1, C_1, D_1, E_1 are complanar. This condition is surely fulfilled if we require for quadruples A_1, B_1, C_1, D_1 and A_1, B_1, C_1, E_1 to be complanar, i.e.,

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ e_1 & e_2 & e_3 & 1 \end{vmatrix} = 0. \qquad (19)$$

First we will search for such k that points A_1, B_1, C_1, D_1 are complanar. We eliminate all variables besides x, y, z, u, v, w, k and get

```
Use R::=Q[xyzuvwa[1..3]b[1..3]c[1..3]d[1..3]e[1..3]k];
I:=Ideal(a[2]-y/2-ky/2,a[3]-z/2-kz/2,b[1]-u/2-k(u/2-x),b[2]-v/2
-kv/2,b[3]-(z+w)/2-k(z+w)/2,c[1]-u/2-ku/2,c[2]-v/2-k(v/2-y),c[3]
-w/2-kw/2,d[1]-x/2-kx/2,d[3]+kz,e[1]-x/2-k(x/2-u),e[2]-y/2-k(y/2
-v),e[3]+kw,-b[1]c[2]d[3]-d[1]b[2]c[3]+b[3]c[2]d[1]+d[3]b[2]c[1]
+d[1]a[2]c[3]-a[3]c[2]d[1]-d[3]a[2]c[1]-d[1]a[2]b[3]+a[3]b[2]d[1]
+d[3]a[2]b[1]+b[1]c[2]a[3]+c[1]a[2]b[3]-a[3]b[2]c[1]-c[3]a[2]b[1]);
Elim(a[1]..e[3],I);
```

the only condition

$$z(5k^2-1)(3xyk-yuk-xvk+xy-yu-xv)=0.$$

In an analogous way for a quadruple A_1, B_1, C_1, E_1 we get

$$x(5k^2-1)(yzk-zvk+2ywk+yz-zv)=0.$$

we see that both conditions (30) (points A_1, B_1, C_1, D_1 and A_1, B_1, C_1, E_1 are complanar) are fulfilled for the values $k=1/\sqrt{5}$ or $k=-1/\sqrt{5}$. For $k=1/\sqrt{5}$ we get a *convex* affine - regular pentagon $A_1B_1C_1D_1E_1$, whereas the value $k=-1/\sqrt{5}$ gives a *non-convex* star affine - regular pentagon $A_1'B_1'C_1'D_1'E_1'$.

Remarks:

1) Both affine-regular pentagons $A_1B_1C_1D_1E_1$ and $A_1'B_1'C_1'D_1'E_1'$ lie in different planes which pass through the common centroid of both affine - regular pentagons and the original pentagon $ABCDE$.

2) Douglas's theorem for a skew pentagon can be generalized to an arbitrary skew n-gon [85], [68], [71].

Geometric inequalities

In this part we will study some geometric inequalities discovered and proved by computer. However we should realize that we are working in an algebraic closed field, in our case in the field of complex numbers which cannot be ordered. Hence we can not use the signs $>$ or $<$.

The method we will use is as follows. A given expression we transform into such a form from which its non-negativity (or non-positivity) would be seen. Most often a given formula is expressed in the form of the sum of squares from which non-negativity follows. However, this is not always possible. In such cases another expression of a formula is necessary as we will see by the investigation of the inequality of Euler.

Parallelogram law

This equality is known as the *parallelogram law* [26]:

Given a parallelogram with the lengths of sides a, b and diagonals e, f. Then

$$2(a^2 + b^2) = e^2 + f^2. \tag{20}$$

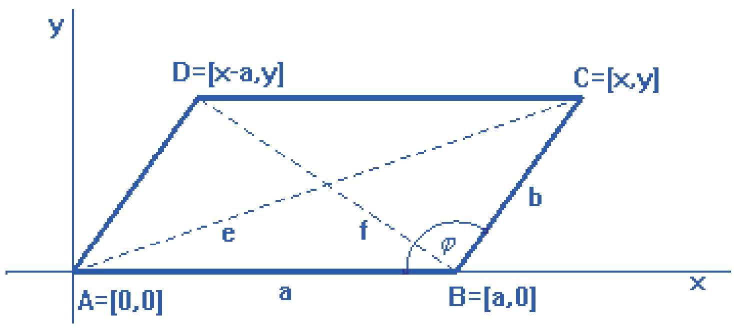

Figure 16: Parallelogram law: $2(a^2 + b^2) = e^2 + f^2$

Remark:

The parallelogram law (20) was already known in a little bit different form to ancient Greeks. It became familiar since the year 1935, when Jordan and von Neumann showed that Banach space in which (20) holds is Hilbert space. □

Inequality between diagonals of a quadrilateral

We will generalize the parallelogram law. It holds [26]:

Given a quadrilateral ABCD with lengths of sides a, b, c, d and diagonals e, f. Then

$$a^2 + b^2 + c^2 + d^2 \geq e^2 + f^2. \qquad (21)$$

The sign of equality in (21) is attained if and only if ABCD is a parallelogram.

A special case of (21) is just the parallelogram law (20). The inequality (21) holds even for a skew quadrilateral $ABCD$ as we will show later.

To prove (21), we will try to express the difference $(a^2 + b^2 + c^2 + d^2) - (e^2 + f^2)$ as a sum of squares, from which non-negativity of $(a^2 + b^2 + c^2 + d^2) - (e^2 + f^2)$ would follow. Denote

$$(a^2 + b^2 + c^2 + d^2) - (e^2 + f^2) = k \qquad (22)$$

and investigate the ideal $J = I \cup \{a^2 + b^2 + c^2 + d^2 - e^2 - f^2 - k\}$, where I is the ideal from the last case. The elimination of dependent variables b, c, d, e, f in the ideal J returns

```
Use R::=Q[xyuvabcdefk];
J:=Ideal((x-a)^2+y^2-b^2,(x-u)^2+(y-v)^2-c^2,u^2+v^2-d^2,x^2+y^2
-e^2,(u-a)^2+v^2-f^2,a^2+b^2+c^2+d^2-e^2-f^2-k);
Elim(b..f,J);
```

the only polynomial $x^2+y^2-2xu+u^2-2yv+v^2-2xa+2ua+a^2-k$, from which we get

$$k = (x - u - a)^2 + (y - v)^2. \qquad (23)$$

The substitution of k from (23) into (22) gives the following identity

$$a^2 + b^2 + c^2 + d^2 - e^2 - f^2 = (x - u - a)^2 + (y - v)^2, \qquad (24)$$

which implies the inequality (21). The equality in (21) is attained if and only if $ABCD$ is a parallelogram as we could see in the previous part.

Euler's inequality

In 1765 L. Euler [21] published the relation (25) which expresses the distance d of the incenter and circumcenter of an arbitrary triangle

$$d = \sqrt{r(r - 2p)}\ , \qquad (25)$$

where r is the circumradius and p is the inradius, Fig. 17.

The following inequality, which is a consequence of (25), is

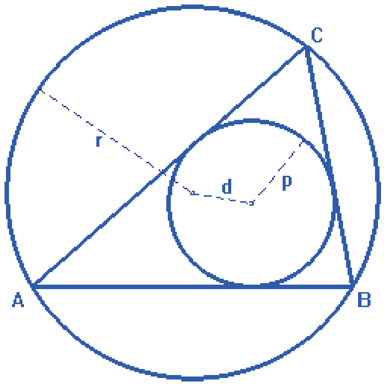

Figure 17: Euler's relation: $d = \sqrt{r(r-2p)}$

$$r \ \geq \ 2p \ , \tag{26}$$

called *Euler's inequality* [10]. It is obvious that the equality in (26) is attained iff a triangle is equilateral, because only in this case the circumcenter and incenter coincide and $d = 0$ in (25).

We will prove the Euler's inequality (26) by computer without introducing a coordinate system, see also W. Koepf [49].

Let ABC be an arbitrary triangle with the lengths of sides a, b, c. Denote by r and p its circumradius and inradius respectively and let f be the area of ABC. For the area f of a triangle ABC we have the following familiar relations:

$$h_1 : f - p(a+b+c)/2 = 0 \ ,$$
$$h_2 : f - abc/(4r) = 0 \ ,$$
$$h_3 : 16f^2 - (a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 0 \ .$$

Suppose that a, b, c, p, r, f are positive real numbers. The main idea of the method is as follows. We will try to express $r - 2p$ in such a form from which it would be clear that $r - 2p$ is greater than or equals zero. That is why we introduce a slack variable k such that $r - 2p = k$, i.e.,

$$h_4 : r - 2p - k \ .$$

In the ideal $I = (h_1, h_2, h_3, h_4)$ we will eliminate p and r to obtain polynomials in variables a, b, c, f, k. We enter

```
Use R::=Q[abcprfk];
I:=Ideal(2f-p(a+b+c),4fr-abc,16f^2-(a+b+c)(-a+b+c)(a-b+c)(a+b-c),
r-2p-k);
Elim(p..r,I);
```

and get a few polynomials from which the following one, after dividing it by a non zero factor f, leads to the equation of the form

$$4fk = a^3 - a^2b - ab^2 + b^3 - a^2c + 3abc - b^2c - ac^2 - bc^2 + c^3. \quad (27)$$

From (27) we see that the expression $k = r - 2p$ is non-negative iff the polynomial on the right in (27) is non-negative. It is easy to show that the equality

$$a^3 - a^2b - ab^2 + b^3 - a^2c + 3abc - b^2c - ac^2 - bc^2 + c^3 =$$

$$= 1/2[(a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2] \quad (28)$$

holds, cf. [49].

The expression on the right hand side in (28) is non-negative. Namely it is the sum of non-negative expressions which consist of squares $(a-b)^2$, $(b-c)^2$, $(c-a)^2$ and expressions $a+b-c$, $b+c-a$, $c+a-b$ which are positive due to the triangle inequality. The equality in (26) occurs if and only if $a = b = c$ in (28) on the right, i.e., iff a triangle ABC is equilateral.

Regular polygons

Before Christmas 1969 two chemists A. Dreiding and J.D. Dunitz visited the well-known mathematician B.L. van der Waerden. The latter talked about fixed and movable forms of cyclic hexane and octane. He also mentioned cyclic pentane and insisted that a skew pentagon with the same lengths of sides and the same angles must necessary lie in a plane. Van der Waerden was surprised at this statement since none to him known theories such a "simple" result mentioned. Also G. Pólya disclaimed any previous knowledge of the theorem and added "if van der Waerden didn't know about it then it wasn't known to mathematics".

A polygon $P_0, P_1, \ldots, P_{n-1}$ whose sides has the same length, i.e., $|P_jP_{j+1}|$ is a constant for all $j = 0, 1, \ldots, n-1$, we call *equilateral.* Similarly an n-gon is *k-equilateral* if $|P_jP_{j+\nu}| = d_\nu$ for all $j = 0, 1, \ldots, n-1$ and $\nu = 1, 2, \ldots, k$, where the constants d_ν are *parameters.* A polygon is called *regular* if for all $\nu = 1, 2, \ldots, n-1$ the lengths of segments $P_jP_{j+\nu}$ are independent of j, or in other words, if a polygon is $(n-1)$-equilateral.
Thus an equilateral n-gon is 1-equilateral with the parameter d_1, 2-equilateral n-gon is equilateral and equiangular with parameters d_1, d_2, etc. If we introduce $d_0 = 0$ then a regular n-gon is characterized by relations

$$|P_jP_{j+\nu}| = d_\nu, \quad \text{for all} \quad j, \nu = 0, 1, \ldots, n-1, \qquad (29)$$

which means that all diagonals of the "same" kind (next but one vertex, next but two vertices,...) have the same length.

Regular pentagon

We will prove the following theorem mentioned above:

A regular skew pentagon $ABCDE$ in the Euclidean space E^3 is given. Then $ABCDE$ is a planar pentagon.

Assume that a pentagon is equilateral with the length of a side a and equiangular with the length of a diagonal u.
We will introduce the Cartesian system of coordinates such that for the vertices A, B, C, D, E of a pentagon $A = [a, 0, 0]$, $B = [b_1, b_2, 0]$, $C = [c_1, c_2, c_3]$, $D = [d_1, d_2, d_3]$, $E = [0, 0, 0]$, Fig. 18. The following

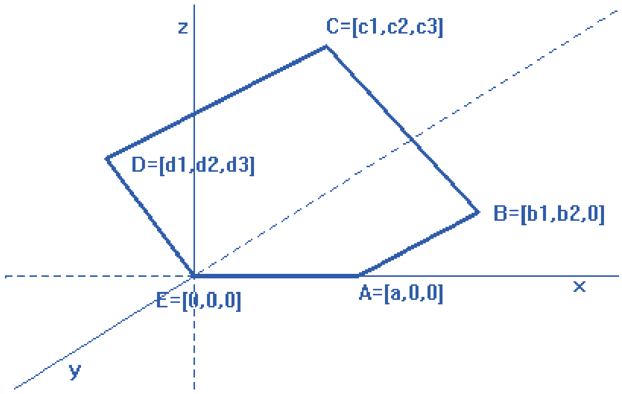

Figure 18: Regular pentagon

relations are fulfilled:

$|AB| = a \Leftrightarrow h_1 : (b_1 - a)^2 + b_2^2 - a^2 = 0,$
$|BC| = a \Leftrightarrow h_2 : (c_1 - b_1)^2 + (c_2 - b_2)^2 + c_3^2 - a^2 = 0,$
$|CD| = a \Leftrightarrow h_3 : (d_1 - c_1)^2 + (d_2 - c_2)^2 + (d_3 - c_3)^2 - a^2 = 0,$
$|DE| = a \Leftrightarrow h_4 : d_1^2 + d_2^2 + d_3^2 - a^2 = 0,$
$|AC| = u \Leftrightarrow h_5 : (c_1 - a)^2 + c_2^2 + c_3^2 - u^2 = 0,$
$|BD| = u \Leftrightarrow h_6 : (d_1 - b_1)^2 + (d_2 - b_2)^2 + d_3^2 - u^2 = 0,$
$|CE| = u \Leftrightarrow h_7 : c_1^2 + c_2^2 + c_3^2 - u^2 = 0,$
$|DA| = u \Leftrightarrow h_8 : (d_1 - a)^2 + d_2^2 + d_3^2 - u^2 = 0,$
$|EB| = u \Leftrightarrow h_9 : b_1^2 + b_2^2 - u^2 = 0.$

Points $ABCDE$ are complanar $\Leftrightarrow$

$$z_1 : \begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ b_1 & b_2 & 0 & 1 \\ c_1 & c_2 & c_3 & 1 \end{vmatrix} = 0 \quad \text{and} \quad z_2 : \begin{vmatrix} 0 & 0 & 0 & 1 \\ a & 0 & 0 & 1 \\ b_1 & b_2 & 0 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0. \qquad (30)$$

We will explore whether both polynomials z_1, z_2 of the conclusion belong to the radical of I. Hence we will find out whether $1 \in J$, where $J = I \cup \{ab_2c_3t - 1\}$ and $I = (h_1, h_2, \ldots, h_9)$.

For z_1 we get

```
Use R::=Q[aub[1..3]c[1..3]d[1..3]st];
J:=Ideal((b[1]-a)^2+b[2]^2-a^2,(c[1]-b[1])^2+(c[2]-b[2])^2+
c[3]^2-a^2,(d[1]-c[1])^2+(d[2]-c[2])^2+(d[3]-c[3])^2-a^2,
d[1]^2+d[2]^2+d[3]^2-a^2,(c[1]-a)^2+c[2]^2+c[3]^2-u^2,(d[1]
-b[1])^2+(d[2]-b[2])^2+d[3]^2-u^2,c[1]^2+c[2]^2+c[3]^2-u^2,
(d[1]-a)^2+d[2]^2+d[3]^2-u^2,b[1]^2+b[2]^2-u^2,ab[2]c[3]t-1);
NF(1,J);
```

that the normal form `NF(1,J)=0`. It means that the points A, B, C, E are complanar. Similarly we will show that the points A, B, D, E are complanar as well. Whence from this our theorem follows - a regular skew pentagon $ABCDE$ is planar.

Remark:

In the last case we examined the normal form `NF(1,J)`, where the ideal J contained the negated conclusion $ab_2c_3t - 1$. The result `NF(1,J)=0` means that the conclusion polynomial ab_2c_3 is an element of the radical $\sqrt{I}$ of I, from which $ab_2c_3 = 0$ follows.
Usually it suffices to find out whether the conclusion polynomial ab_2c_3 belongs to the ideal I. Let us do it.

We get

```
Use R::=Q[aub[1..3]c[1..3]d[1..3]st];
I:=Ideal((b[1]-a)^2+b[2]^2-a^2,(c[1]-b[1])^2+(c[2]-b[2])^2+
c[3]^2-a^2,(d[1]-c[1])^2+(d[2]-c[2])^2+(d[3]-c[3])^2-a^2,
d[1]^2+d[2]^2+d[3]^2-a^2,(c[1]-a)^2+c[2]^2+c[3]^2-u^2,(d[1]
-b[1])^2+(d[2]-b[2])^2+d[3]^2-u^2,c[1]^2+c[2]^2+c[3]^2-u^2,
(d[1]-a)^2+d[2]^2+d[3]^2-u^2,b[1]^2+b[2]^2-u^2);
NF(ab[2]c[3],I);
```

the result ab_2c_3 which is not zero. Hence the polynomial ab_2c_3 *does not* belong to the ideal I ! However ab_2c_3 belongs to the radical $\sqrt{I}$ of I, i.e., there exists a natural number m such that $(ab_2c_3)^m$ belongs to the ideal I. It is easy to verify that in this case $m = 3$, i.e., $(ab_2c_3)^3 \in I$.

S.Ch. Chou in [39] on the page 78 writes that "for all theorems we have found in practice $I = \sqrt{I}$". However in our case I is a *proper* subset of its radical. The polynomial ab_2c_3 *is not* an element of ideal I, but it belongs to the radical $\sqrt{I}$ of I. Thus, this case is an example of a problem in which it is necessary to use the radical $\sqrt{I}$ of I instead of the ideal I to prove a statement. □

To give another proof we will apply the following formula for the volume of a simplex.
The volume V_n of a simplex $A_1, A_2, \ldots, A_{n+1}$ in E^n can be expressed in terms of all mutual distances $|A_iA_j| = a_{ij}$ between vertices of a simplex in the form of the so called Cayley - Menger's determinant.

Denoting $a_{ij}^2 = d_{ij}$ then:

$$(-1)^{n+1} n^2 (n!)^2 V_n^2 = D_n = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{12} & d_{13} & \dots & d_{1,n+1} \\ 1 & d_{21} & 0 & d_{23} & \dots & d_{2,n+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & d_{n+1,1} & \dots & \dots & \dots & 0 \end{vmatrix}. \quad (31)$$

If we put $V_n = 0$ then by (31) also $D_n = 0$ and a simplex $A_1, A_2, \dots, A_{n+1}$ can be placed into a space whose dimension is less than n.

Regular heptagon

The case of a regular pentagon from the previous part is well-known. Let us look at a regular heptagon and its existence in spaces of various dimensions. Let $A_1A_2A_3A_4A_5A_6A_7$ be a regular heptagon. In accor-

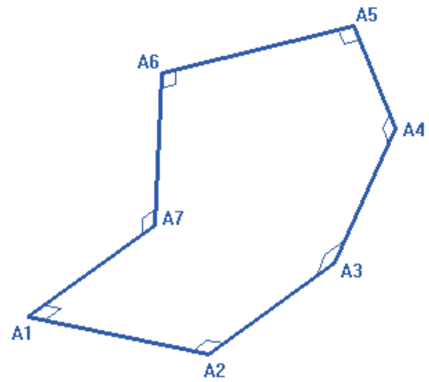

Figure 19: The example of an equilateral and equiangular heptagon which is not regular

dance with the definition a regular heptagon is 3-equilateral.

In a regular heptagon $A_1A_2A_3A_4A_5A_6A_7$ let us denote

$|A_1A_2| = |A_2A_3| = |A_3A_4| = |A_4A_5| = |A_5A_6| = |A_6A_7| = |A_7A_1| = a,$
$|A_1A_3| = |A_3A_5| = |A_5A_7| = |A_7A_2| = |A_2A_4| = |A_4A_6| = |A_6A_1| = b,$
$|A_1A_4| = |A_4A_7| = |A_7A_3| = |A_3A_6| = |A_6A_2| = |A_2A_5| = |A_5A_1| = c.$

Consider a heptagon in a six dimensional Euclidean space E^6 which is a space of minimal dimension in which an arbitrary heptagon can always be placed.

By the formula (31) for the volume of a simplex $A_1A_2A_3A_4A_5A_6\ A_7$

in E^6 it holds

$$D_6 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 & c^2 & b^2 & a^2 \\ 1 & a^2 & 0 & a^2 & b^2 & c^2 & c^2 & b^2 \\ 1 & b^2 & a^2 & 0 & a^2 & b^2 & c^2 & c^2 \\ 1 & c^2 & b^2 & a^2 & 0 & a^2 & b^2 & c^2 \\ 1 & c^2 & c^2 & b^2 & a^2 & 0 & a^2 & b^2 \\ 1 & b^2 & c^2 & c^2 & b^2 & a^2 & 0 & a^2 \\ 1 & a^2 & b^2 & c^2 & c^2 & b^2 & a^2 & 0 \end{vmatrix} = \tag{32}$$

$$= -7(a^6+3a^4b^2-4a^2b^4+b^6-4a^4c^2-a^2b^2c^2+3b^4c^2+3a^2c^4-4b^2c^4+c^6)^2.$$

Now we express the volume of a simplex which is determined by the points A_1, A_2, A_3, A_4, A_5, A_6 in such a way that we delete the last row and column in the previous determinant. The result is

$D_5 = 2(2a^4 - 3a^2b^2 + 2b^4 - 3a^2c^2 - 3b^2c^2 + 2c^4)(a^6 + 3a^4b^2 - 4a^2b^4 + b^6 - 4a^4c^2 - a^2b^2c^2 + 3b^4c^2 + 3a^2c^4 - 4b^2c^4 + c^6)$.

The comparison of the determinants D_6 and D_5 gives that $D_6 = 0$ implies $D_5 = 0$. Hence if the dimension of a regular heptagon is not six then it is neither five. Therefore it must lie in the space of dimension four or less.

Let us proceed further and express the volume of a simplex which is given by the points A_1, A_2, A_3, A_4, A_5. The result is

$$D_4 = -(a^4-12a^2b^2+8b^4+2a^2c^2-5b^2c^2+c^4)(a^2-bc-c^2)(a^2+bc-c^2).$$

Let us put $D_4 = 0$ which means that a simplex $A_1A_2A_3A_4A_5$ is of dimension three or less, and investigate the volume of a tetrahedron $A_1A_2A_3A_4$ in E^3. We obtain

$$D_3 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & a^2 & b^2 \\ 1 & b^2 & a^2 & 0 & a^2 \\ 1 & c^2 & b^2 & a^2 & 0 \end{vmatrix} = \tag{33}$$

$$= -2(a^2 - b^2 + ac)(a^2 - b^2 - ac)(a^2 + 2b^2 - c^2).$$

If the condition $D_4 = 0$ (assuming that also $D_6 = 0$) implies $D_3 = 0$ then the points A_1, A_2, A_3, A_4 are complanar.

However at the first sight the determinants D_4 and D_3 do not contain like factors, and so it could seem that the implication mentioned above does not hold.

Suppose that for a regular heptagon $A_1A_2A_3A_4A_5A_6A_7$ relations $D_6 = 0$ and $D_4 = 0$ hold. We will explore whether then $D_3 = 0$ follows. Hence given the ideal $I = (D_6, D_4)$ we are asking whether D_3 belongs to the radical $\sqrt{I}$ of I. We enter

```
Use R::=Q[abct];
I:=Ideal(a^6+3a^4b^2-4a^2b^4+b^6-4a^4c^2-a^2b^2c^2+3b^4c^2+3a^2c^4
-4b^2c^4+c^6,(a^4-12a^2b^2+ 8b^4+2a^2c^2- 5b^2c^2+c^4)(a^2-bc-c^2)
(a^2+bc-c^2),(a^2-b^2+ac)(a^2-b^2-ac)(a^2+2b^2-c^2)t-1);
NF(1,I);
```

and obtain the result `NF(1,I)=0`.

We proved the theorem:

A regular heptagon lies either in the Euclidean space E^6 or in E^4 or i E^2, i.e., its dimension is either 6 or 4 or 2.

From the equations $D_4 = 0$ and $D_3 = 0$ we get relations which enable to compute lengths of sides and diagonals a, b, c of a regular heptagon. The system of equations

$$a^2 + ac - b^2 = 0, \qquad a^2 + bc - c^2 = 0 \tag{34}$$

characterizes a convex regular heptagon, Fig. 20.

Similarly we will derive the equations for remaining two non-convex

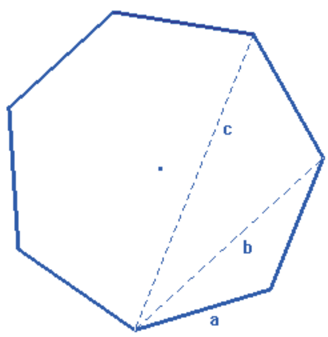

Figure 20: Convex regular heptagon

star regular heptagons, Fig. 21. For a heptagon in Fig. 21 on the left (so called 2-regular heptagon) it holds

$$a^2 - bc - c^2 = 0, \qquad ab - b^2 + c^2 = 0$$

and for a heptagon in Fig. 21 on the right (so called 3-regular heptagon)

$$ab + b^2 - c^2 = 0, \qquad a^2 - ac - b^2 = 0.$$

Remarks:

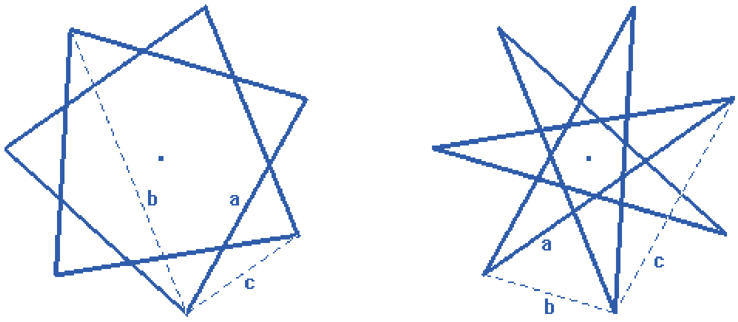

Figure 21: Non - convex regular heptagons

1) In the last computation we showed that the polynomial D_3 is an element of the radical $\sqrt{I}$ of I. If we investigate whether the polynomial D_3 is an element of the ideal I instead of its radical $\sqrt{I}$, we get

```
Use R::=Q[abct];
I:=Ideal(a^6+3a^4b^2-4a^2b^4+b^6-4a^4c^2-a^2b^2c^2+3b^4c^2
+3a^2c^4-4b^2c^4+c^6,(a^4-12a^2b^2+8b^4+2a^2c^2-5b^2c^2+c^4)
(a^2-bc-c^2)(a^2+bc-c^2));
NF((a^2-b^2+ac)(a^2-b^2-ac)(a^2+2b^2-c^2),I);
```

the result $-3a^4b^2 + a^2b^4 + b^6 + 2a^4c^2 + a^2b^2c^2 - 4b^4c^2 - 2a^2c^4 + 4b^2c^4 - c^6$ which is not zero.

Hence the polynomial $D_3 = (a^2 - b^2 + ac)(a^2 - b^2 - ac)(a^2 + 2b^2 - c^2)$ *is not* an element of the ideal I. We again encounter the case that it does not suffice to explore whether the given polynomial belongs to the ideal I, but it is necessary to examine its existence in the radical $\sqrt{I}$ of I. A close inspection shows that the polynomial D_3^3 belongs to the ideal I.

2) The isoperimetric inequality for n-gons in the plane says [6] that *from all n-gons of the given perimeter, a regular n-gon has the greatest area.* A spatial analog of the isoperimetic inequality is as follows [9]:

From all skew n-gons in E^d with the given perimeter find an n-gon of the maximal volume of its convex hull.

This spatial analog of the isoperimetric inequality for n-gons was

solved in spaces of *even* dimension d, where extremal n-gons are more-dimensional analogs of regular polygons in the plane [64].
In the space of *odd* dimension d this problem has not been solved yet. Several special cases in E^3 for a skew quadrilateral [57], pentagon and hexagon [47], [48] have been solved. Especially an extremal pentagon is of interest since by the theory above it *can not* be regular. Hence a natural question arises, how do extremal n-gons in E^3 $n \geq 7$ look like? □

Miscellaneous

Non - elementary constructions

Due to modern tools like computers, mathematical software and the theory of automatic proving theorems we are able to solve even non-elementary constructions. An example of such a non - elementary construction is the following problem, see [43]:

Given four lines a, b, c, d in the plane. Construct a square $KLMN$ such that $K \in a$, $L \in b$, $M \in c$, $N \in d$.

Let us choose the Cartesian coordinate system so that the vertices K, L, M, N of a square have coordinates $K = [k_1, k_2]$, $L = [l_1, l_2]$, $M = [m_1, m_2]$, $N = [n_1, n_2]$. Let the lines a, b, c, d have equations:

$a : a_1x + a_2y + a_3 = 0$, $b : b_1x + b_2y + b_3 = 0$, $c : c_1x + c_2y + c_3 = 0$, $d : d_1x + d_2y + d_3 = 0$.

Then

$K \in a \Leftrightarrow h_1 : a_1k_1 + a_2k_2 + a_3 = 0,$
$L \in b \Leftrightarrow h_2 : b_1l_1 + b_2l_2 + b_3 = 0,$
$M \in c \Leftrightarrow h_3 : c_1m_1 + c_2m_2 + c_3 = 0,$
$N \in d \Leftrightarrow h_4 : d_1n_1 + d_2n_2 + d_3 = 0.$

To ensure that $KLMN$ is a square, whose vertices K, L, M, N lie on the lines a, b, c, d respectively, we will rotate the vector $L - K$ by 90° in a positive sense. We get the vector $N - K$ which we denote by $\text{rot}(L - K) = N - K$. Then we will rotate the vector $K - N$ by 90° in the same sense and get the vector $M - N$, etc., see Fig. 22. The

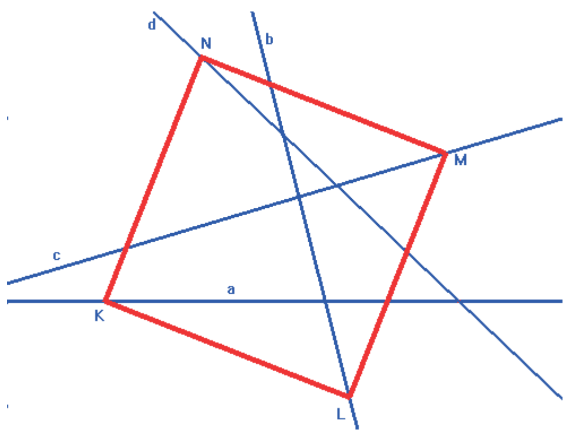

Figure 22: A square $KLMN$ whose vertices lie on the given lines a, b, c, d respectively

elimination gives the result

$$k_1 = \frac{b_2c_1 + b_2c_2 + c_1d_1 + c_2d_1 - b_2d_2 - c_1d_2 + c_2d_2 + c_1 - c_2 - d_1}{b_2c_1d_1 + b_2c_2d_1 - b_2c_1d_2 + c_1d_1 - c_1d_2 + c_2d_2}.$$

Similarly we will find the remaining unknowns. One solution we can see in Fig. 22. From the construction it is seen that there exist at most four squares $KLMN$ with the given properties. □

Locus of points of given properties

In this chapter we will investigate a locus of points of given properties by the method of elimination of variables. See also [102]. We will solve the following problem:

On the sides BC, CA, AB of a triangle ABC points D, E, F which divide the sides of ABC respectively in the same ratio k, i.e.,

$$k = (BCD) = (CAE) = (ABF),$$

are given. Determine the locus of circumcenters of triangles DEF by varying the value k.

We get the cubic curve.

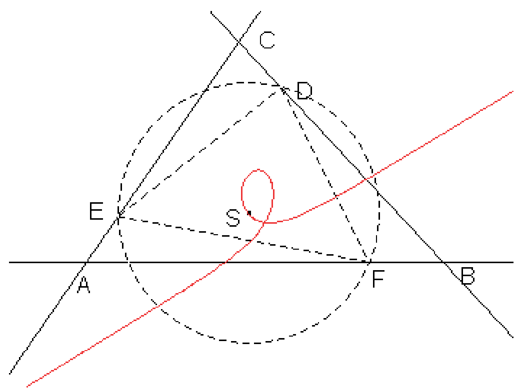

Figure 23: Curve of 3rd degree as the locus of points of the given property

In a special case for $a = 2, b = 0, c = 2$ we get the curve

$$x^3 + y^3 - 3x^2 + xy - 3y^2 + 2x + 2y - 1 = 0, \qquad (35)$$

which has the only singular point $[1, 1]$. Substitution $x' = x + 1$, $y' = y + 1$ into (35) leads to the equation

$$x'^3 + x'y' + y'^3 = 0$$

which is the equation of the well-known *Descartes' folium.*

Theorem of Viviani

Theorem of Viviani [8] reads, Fig. 24:

Given an equilateral triangle ABC and an arbitrary point P inside ABC or on its sides. Then the sum of distances of P to the sides of ABC is constant and equals to the length h of the altitude of ABC, i.e.,

$$|PK| + |PL| + |PM| = h, \tag{36}$$

where K, L, M are the feet of perpendiculars from P to the sides BC, CA, AB respectively.

In this case we will change the order of proofs methods and first we

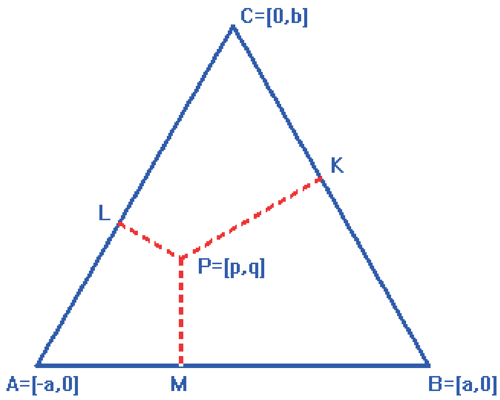

Figure 24: Theorem of Viviani: The sum of distances of a point P to the sides of an equilateral triangle ABC is constant

start with a classical proof. We will show an elegant proof which is from [8], see also [60].

Consider an equilateral triangle ABC and its shifted congruent image $A'B'C'$ such that the side $B'C'$ passes through the point P, Fig. 25. Then

$h = |C'Q| + |PM| = |SP| + |PM| = |SL| + |LP| + |PM| = |PK| + |PL| + |PM|$.

□

Now we will explore Viviani's theorem by computer. We will try to derive the theorem. In addition we will succeed even more - we will generalize the theorem.

Choose the Cartesian system of coordinates so that $A = [-a, 0]$, $B = [a, 0]$, $C = [0, b]$, $P = [p, q]$, $K = [k_1, k_2]$, $L = [l_1, l_2]$, $M = [m, 0]$, Fig. 24. We have:

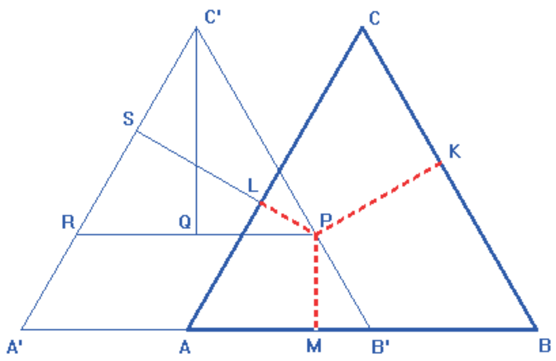

Figure 25: Classical proof of theorem of Viviani: The sum of distances of a point P to the sides of an equilateral triangle ABC is constant

$PK \perp BC \Leftrightarrow h_1 : a(p - k_1) - b(q - k_2) = 0,$
$K \in BC \Leftrightarrow h_2 : ab - bk_1 - ak_2 = 0,$
$PL \perp AC \Leftrightarrow h_3 : a(p - l_1) + b(q - l_2),$
$L \in AC \Leftrightarrow h_4 : bl_1 - al_2 + ab,$
$PM \perp AB \Leftrightarrow h_5 : a(p - m) = 0.$

Denote $|PK| = u, |PL| = v, |PM| = w$. Then

$|PK| = u \Leftrightarrow h_6 : (p - k_1)^2 + (q - k_2)^2 - u^2,$
$|PL| = v \Leftrightarrow h_7 : (p - l_1)^2 + (q - l_2)^2 - v^2,$
$|PM| = w \Leftrightarrow h_8 : (p - m)^2 + q^2 - w^2.$

An equilateral triangle ABC with the length of a side $2a$ and height b fulfils

$|AB| = |BC| = |CA| = 2a \Leftrightarrow h_9 : 3a^2 - b^2 = 0.$

To derive (36) we will search for a relation between distances u, v, w and the length of the side of ABC. In the ideal $I = (h_1, h_2, \ldots, h_9)$ we eliminate all variables besides a, u, v, w and get

```
Use R::=Q[auvwpqbk[1..2]l[1..2]m];
I:=Ideal(a(p-k[1])-b(q-k[2]),ab-bk[1]-ak[2],a(p-l[1])+b(q-
l[2]),bl[1]-al[2]+ab,a(p-m),(p-k[1])^2+(q-k[2])^2-u^2,(p-
l[1])^2+(q-l[2])^2-v^2,(p-m)^2+q^2-w^2,3a^2-b^2); Elim(p..m,I);
```

the equation

$$a^2((-u+v+w)^2-3a^2)((u-v+w)^2-3a^2)((u+v-w)^2-3a^2)((u+v+w)^2-3a^2) = 0. \tag{37}$$

In this decomposition the expression $(u + v + w)^2 - 3a^2$ occurs, which is equivalent to $|u+v+w| = a\sqrt{3}$ which is (36). However there are also other possibilities. What is the geometric meaning of the equality (37)? The equation $a = 0$ means that all the three vertices coincide. We will exclude this condition. Conditions $(-u+v+w)^2 - 3a^2 = 0$, $(u-v+w)^2 - 3a^2 = 0$, $(u+v-w)^2 - 3a^2 = 0$ express the fact that the "sum" of distances of a point P to the sides of a triangle ABC is constant also in the case when a point P is outside of a triangle ABC. To express the given fact it is advantageous to introduce the signed distance (or algebraic sum) of a point to a straight line. The *signed distance* of a point P to a line AB is its distance to a line AB with the sign + if the sense of a motion on the perimeter of a triangle from P to A and B is counter-clockwise, i.e., it is positive. In the opposite case the signed distance has the sign $-$. We will denote the signed distance of a point P to a line AB by $\|P, AB\|$. Hence, for instance $\|P, AB\| = -\|P, BA\|$. We derived the following generalization of the

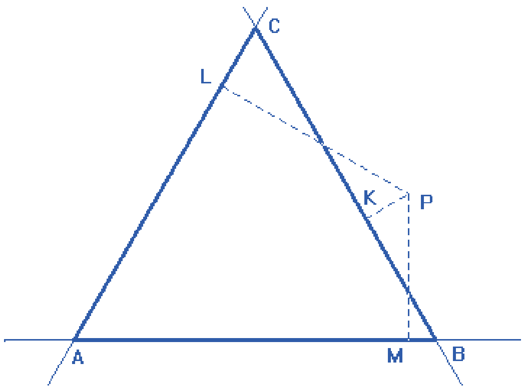

Figure 26: $\|P, AB\| + \|P, BC\| + \|P, CA\| = u - v + w = h$.

theorem of Viviani for an arbitrary point of the plane of a triangle, Fig. 26.:

Given an equilateral triangle ABC and an arbitrary point P in the plane of a triangle. Then the sum of signed distances of a point P to the sides of a triangle obeys

$$\|P, AB\| + \|P, BC\| + \|P, CA\| = h, \tag{38}$$

where h is the length of the altitude of ABC.

Let us add a classical proof of (38). For the area $|ABC|$ of an equilateral triangle ABC we have

$$|ABC| = \|PAB\| + \|PBC\| + \|PCA\|, \tag{39}$$

where $\|PAB\|, \|PBC\|, \|PCA\|$ are signed areas of triangles PAB, PBC, PCA. Assume that the length of the side of ABC equals $2a$. Using the formula (39) we get

$$ah = a\|P, AB\| + a\|P, BC\| + a\|P, CA\|.$$

From here (38) follows. □

Gauss's line

The following problem is known under "The theorem on a complete quadrilateral" or "Gauss's line", see [5], [39] [93], [50], [101].

The three midpoints of diagonals of a complete quadrilateral are collinear.

The line which contains all three midpoints is called the *Gauss's line*, Fig. 27. Choose the affine coordinate system such that

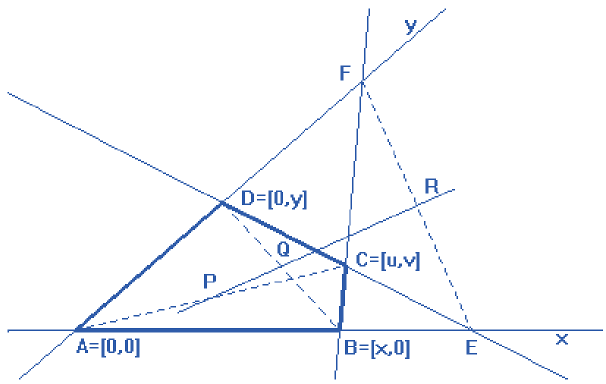

Figure 27: Gauss's line - a computer proof

$A = [0, 0]$, $B = [x, 0]$, $C = [u, v]$,D=[0,y],E=[e,0],$F = [0, f]$, $P = [p_1, p_2]$, $Q = [q_1, q_2]$, $R = [r_1, r_2]$.

Then the following relations hold:

$E \in AB \cap CD \Leftrightarrow h_1 : ev + uy - ey = 0,$
$F \in AD \cap BC \Leftrightarrow h_2 : fu + vx - fx = 0,$
P je sted $AC \Leftrightarrow h_3 : 2p_1 - u = 0 \wedge h_4 : 2p_2 - v = 0,$
Q je sted $BD \Leftrightarrow h_5 : 2q_1 - x = 0 \wedge h_6 : 2q_2 - y = 0,$
R je sted $EF \Leftrightarrow h_7 : 2r_1 - e = 0 \wedge h_8 : 2r_2 - f = 0.$

The conclusion z of the statement has the form

PQR are collinear $\Leftrightarrow z : p_1q_2 + q_1r_2 + r_1p_2 - r_1q_2 - r_2p_1 - p_2q_1 = 0.$

We get

```
Use R::=Q[xyuvefp[1..2]q[1..2]r[1..2]];
I:=Ideal(ev+uy-ey,fu+vx-fx,2p[1]-u,2p[2]-v,2q[1]-x,2q[2]-y,
2r[1]-e,2r[2]-f);
NF(p[1]q[2]+q[1]r[2]+r[1]p[2]-r[1]q[2]-r[2]p[1]-p[2]q[1],I);
```

`NF(z,I)=0`. Hence the normal form of the polynomial of the conclusion z equals zero and the statement is proved. □

Let us show a classical proof of the above statement which is by [50] ascribed to K.F. Gauss.

We will prove this theorem be means of the theorem of Menelaus, see [93]. Consider a triangle ADE and let P, Q, R be the midpoints of its sides DE, AD and EA respectively, Fig. 28. We can see that

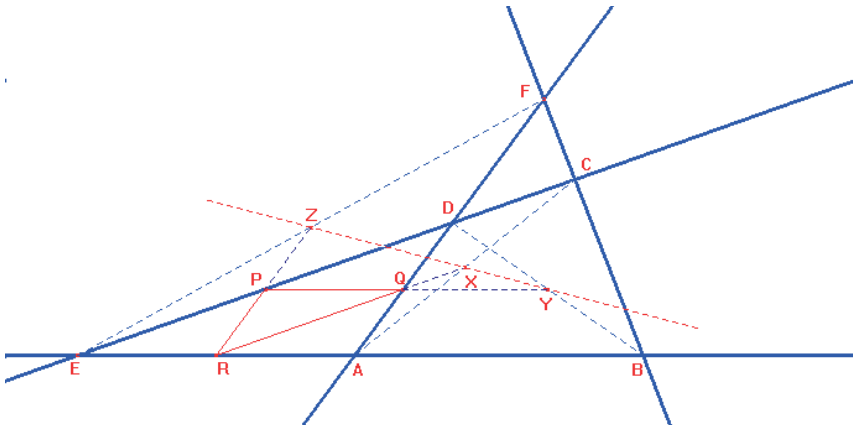

Figure 28: Gauss's line - a classical proof

$PR \parallel AD$ and $Z \in PR$. Analogously $QR \parallel DE$ and $X \in QR$ and $QP \parallel EA$ and $Y \in QP$. In order to prove that the points X, Y, Z are collinear it suffices to verify by the theorem of Menelaus that

$$\frac{\|RX\|}{\|XQ\|} \cdot \frac{\|QY\|}{\|YP\|} \cdot \frac{\|PZ\|}{\|ZR\|} = -1. \tag{40}$$

But

$$\frac{\|RX\|}{\|XQ\|} = \frac{\|EC\|}{\|CD\|}, \ \frac{\|QY\|}{\|YP\|} = \frac{\|AB\|}{\|BE\|}, \ \frac{\|PZ\|}{\|ZR\|} = \frac{\|DF\|}{\|FA\|} \tag{41}$$

and from here

$$\frac{\|RX\|}{\|XQ\|} \cdot \frac{\|QY\|}{\|YP\|} \cdot \frac{\|PZ\|}{\|ZR\|} = \frac{\|EC\|}{\|CD\|} \cdot \frac{\|AB\|}{\|BE\|} \cdot \frac{\|DF\|}{\|FA\|} = -1 \tag{42}$$

since the points C, B, F are collinear. □

Conclusions

In this text the application of the theory of automatic proving, automatic deriving and automatic discovering theorems on a few geometric stories of elementary geometry was given. On many examples, we could see the core of this theory which, especially in last years, due to increasing power of computers and new efficient algorithms becomes meaningful.

The solutions of almost all examples are accompanied by classical ones. The author's effort was to stress that not only classical but also computer solutions contain ideas which show the beauty of both geometry and algebra. In any case we should not prefer only computer or only classical way of solution. Each of both methods has its own strengths and weaknesses. So it is not possible to say that one method is better than the other. Both methods should complement each other.

There were also cases in which a classical method was not available or at least it was not known to the author. In such cases, we had to rely on some of the mentioned computer methods. It is a challenge for us to attempt to find a classical solution in these instances as well. For example, this is the case of a generalization of the formula of Heron for a cyclic pentagon and another n-gons for higher n.
There also exist cases when we are able to solve a problem by a classical method and on the contrary a computer way fails. This is one more challenge - to develop the theory of automatic theorem proving so that it always terminates, i.e., that we are always able to say that a given statement is true or not.

Some problems required a big computational memory cost. It does not make it possible to solve problems which are more complex. An example is the generalization of the formula of Heron for a cyclic n-gons for $n \geq 6$, which we are not able to solve by the used methods. One more challenge is to apply such solution strategies which are not so time consuming. It would enable us to solve much a wider range of problems.

A certain development will be achieved by enhancing the efficiency of computers, but the major and substantial progress in solving problems can be expected with better and more sophisticated algorithms and more ingenious ways of solutions to problems.

Besides the two mentioned basic methods of proving - classical and computer methods - the book should bring to the readers, as the author firmly hopes, a joy and benefit through many known or less known results in geometry. Hence even the reader who does not like computer methods should have a benefit as well. In such a case it suffices to concentrate on classical solutions.

References

[1] M.H. van Aubel: *Note concernant les centres des carres construits sur les côtes d'un polygone quelconque.* Nouv. Corresp. Math. **4** (1878), 40-44.

[2] A. Auric: *Question 3867.* Intermed. Math. **18** (1911), 122.

[3] A. Barlotti: *Una proprieta degli n-agoni che si ottengono transformando in una affinita un n-agono regolare.* Bol. Unione Math. Ital. **3** (1955), 96-98.

[4] H.J. Bartsch: *Matematické vzorce.* SNTL, Praha 1983.

[5] M. Berger: *Geometry I.* Springer - Verlag, Berlin - Heidelberg 1987.

[6] W. Blaschke: *Kreis und Kugel.* Walter de Gruyter & Co, Berlin 1956.

[7] F. van der Blij: *Regular Polygons in Euclidean Space.* Lin. Algebra and Appl. **226-228** (1995), 345-352.

[8] A. Bogomolny: http://www.cut-the-knot.org/geometry.shtml

[9] T. Bonnesen, W. Fenchel: *Theorie der konvexen Körper.* Springer, Berlin 1934.

[10] O. Bottema: *Hoofstukken uit de elementaire meetkunde.* Servire, Den Haag, 1944, Epsilon, Utrecht 1987.

[11] O. Bottema *et al*: *Geometric inequalities.* Groningen 1969.

[12] O. Bottema: *Pentagons with equal sides and equal angles.* Geom. Dedicata **2** (1973), 189-191.

[13] B. Buchberger: *Groebner bases: an algorithmic method in polynomial ideal theory.* In: Multidimensional Systems Theory, ed. N.K. Bose, Reidel, Dordrecht 1985, 184-232.

[14] B. Bydžovský: *Úvod do algebraické geometrie.* Jednota českých matematiků a fyziků, Praha 1948.

[15] D. Cox, J. Little, D. O'Shea: *Ideals, Varieties, and Algorithms.* Second Edition, Springer 1997.

[16] H.S.M. Coxeter, S.L. Greitzer: *Geometry revisited.* Toronto - New York 1967.

[17] B.H. Dörrie: *Triumph der Mathematik.* Breslau 1933.

[18] J. Douglas: *A theorem on skew pentagons.* Scripta Math. **25** (1960), 5-9.

[19] J.D. Dunitz, J. Waser: *The Planarity of the Equilateral, Isogonal Pentagon.* Elem. der Math. **27** (1972), 25-32.

[20] M. Ernestová: *Soustavy algebraických rovnic.* Učitel matematiky **10** (2002), 193-208.

[21] L. Euleri: *Novi commentarii academicae scientiarum Petropolitanae* **11** (1765), 1767, 103-123.

[22] L. Euler: *Geometrica et sphaerica quedam.* Mmoires de l'Acadmie des Sciences de St. Peterbourgh **5** (1812), 96-114.

[23] R.L. Finney: *Dynamic proofs of Euclidean theorems.* Math. Mag. **43** (1970), 177-186.

[24] W. Fischer: *Ein geometrischer Satz.* Arch. Math. Phys. **40** (1863), 460-462.

[25] H. Forder: *The Calculus of Extension.* Chelsea, New York, 1960.

[26] L. Gerber: *Napoleon's theorem and the parallelogram inequality for affine - regular polygons.* Amer. Math. Monthly **87** (1980), 644-648.

[27] O. Giering: *Affine and Projective Generalization of Wallace Lines.* J. Geom. and Graphics **1** (1997), 119-133.

[28] Stephen B. Gray: *Generalizing the Petr - Douglas - Neumann Theorem on N-gons.* Amer. Math. Monthly **110** (2003), 210-227.

[29] B. Grünbaum: *Polygons.* In: The Geometry of Metric and Linear Spaces, Lecture Notes in Mathematics **490** (1975), 147-184.

[30] B. Grünbaum: *Cyclic ratio sums and products.* Crux Math. **24** (1998), 20-25.

[31] B. Grünbaum, G.C. Shephard : *Ceva, Menelaus, and the Area Principle.* Math. Magazine **68** (1995), 254-268.

[32] B. Grünbaum, G.C. Shephard : *Ceva, Menelaus, and Self-transversality.* Geometriae Dedicata **65** (1997), 179-192.

[33] M. de Guzmán: *An Extension of the Wallace - Simson Theorem: Projecting in Arbitrary Directions.* Amer. Math. Monthly **106** (1999), 574-580.

[34] H. Hadwiger, P. Finsler: *Einige Relationen im Dreieck.* Comment. Math. Helv. **10** (1937/38), 316-326.

[35] H. Hadwiger: *Altes und neues über konvexe Körper.* Birkhäuser Verlag, 1955.

[36] D. Hilbert, S. Cohn-Vossen: *Anschauliche Geometrie.* Berlin 1932.

[37] J. Hora: *O některých otázkách souvisejících s využíváním programů počítačové algebry ve škole, II. díl*, Pedagogické centrum, Plzeň 1998.

[38] J. Hora, P. Pech: *Using computer to discover some theorems in geometry.* Acta Acad. Paed. Agriensis **29** (2002), 67-75.

[39] S.C. Chou: *Mechanical Geometry Theorem Proving.* D. Reidel Publishing Company, Dordrecht 1987.

[40] H. Irminger: *Zu einem Satz über räumliche Fünfecke.* Elem. der Math. **25** (1970), 135-136.

[41] B.A. Jefremovič, Ju.S. Iljaščenko: *Pravil'nyje mnogougol'niky.* Vestnik Mosk. Univ. **17** (1962), 18-24.

[42] D. Kapur: *A Refutational Approach to Geometry Theorem Proving.* Artificial Intelligence Journal **37** (1988), 61-93.

[43] A. Karger: *Classical Geometry and Computers.* Journal for Geometry and Graphics **2** (1998), 7-15.

[44] F. Kárteszi: *Contributo al pentagono equilatero ed isogonale.* Ann. Univ. Sci. Budapest, Roland Eötvös Sect. **16** (1973), 63-64.

[45] L. Kiepert: *Solution de question 864.* Nou. Ann. Math. **8** (1869), 40-42.

[46] J.S. Kline, D.J. Velleman: *Yet another proof of Routh's theorem.* Crux. Math. **21** (1995), 37-40.

[47] M. Kočandrlová: *The Isoperimetric Inequality for a Pentagon in E^3 and Its Generalization in E^n.* Čas. pro pěst. mat. **107** (1982), 167-174.

[48] M. Kočandrlová: *Isoperimetrische Ungleichung für geschlossene Raumsechsecke.* Čas. pro pěst. mat. **108** (1983), 248-257.

[49] W. Koepf: *Gröbner Bases and Triangles.* International Journal of Computer Algebra in Mathematics Education **4** (1997), 371-386.

[50] A.G. Konforovič: *Významné matematické úlohy.* SPN, Praha 1989.

[51] G. Kowalewski: *Einführung in die Determinantentheorie.* Veit& Comp., Leipzig 1909.

[52] B. Kutzler: *Careful algebraic translations of geometry theorems.* In: Proceedings ISSAC '89, Portland, ACM Press, New York 1989, 254-263.

[53] J. Lawrence: *K-equilateral $(2k+1)$-gons span only even-dimensional spaces.* In: The Geometry of Metric and Linear Spaces, Lecture Notes in Mathematics **490** (1975), 185-186.

[54] W. Lüssy, E. Trost: *Zu einem Satz über räumliche Fünfecke.* Elem. der Math. **25** (1970), 82-83.

[55] H. Martini: *Neuere Ergebnisse der Elementargeometrie.* In: O. Giering, J. Hoschek (eds): Geometrie und Ihre Anwendungen. Carl Hanser Verlag, München und Wien 1994, 9-42.

[56] H. Martini: *On the theorem of Napoleon and related topics.* Math. Semesterber. **43** (1996), 47-64.

[57] B. Míšek: *O $(n+1)$-úhelníku v E^n s maximálním objemem konvexního obalu.* Čas. pro pěst. mat. **84** (1959), 93-103.

[58] J. Naas, H.L. Schmid: *Mathematisches Wörterbuch.* Berlin - Leipzig 1967.

[59] Z. Nádeník: *Náměty k středoškolské geometrii.* Pokroky matematiky, fyziky a astronomie **19**, (1974).

[60] R. Nelsen: *Proofs Without Words.* MAA, 1993.

[61] R. Nelsen: *Proofs Without WordsII.* MAA, 2000.

[62] B.H. Neumann: *Some remarks on polygons.* J. London Math. Soc. **16** (1941), 230-245.

[63] S.I. Novoselov: *Special'nyj kurs trigonometrii.* Nauka, Moskva 1953.

[64] A.A. Nudel'man: *Izoperimetričeskije zadači dlja vypuklych oboloček lomanych i krivych v mnogomernych prostranstvach.* Mat. Sbornik **96** (1975), 294-313.

[65] I. Pak: *The area of cyclic polygons: Recent progress on Robbins' Conjectures.* Mini survey of 5 pages, to appear in Adv. Applied Math. (Special issue on behalf of David Robbins).

[66] P. Pech: *O Petrově větě.* Rozhledy matem. - fyzikální **67** (1988), 9-15.

[67] P. Pech: *Inequality between sides and diagonals of a space n-gon and its integral analog.* Čas. pro pěst. mat. **115** (1990), 343-350.

[68] P. Pech: *The harmonic analysis of polygons and Napolepon's theorem.* J. Geometry and Graphics **5** (2001), 13-22.

[69] P. Pech: *Computations of the Area and Radius of Cyclic Polygons Given by the Lengths of Sides.* ADG 2004, Lecture Notes in Artificial Intelligence 3763, New York - Heidelberg, Springer 2006, 44-58.

[70] P. Pech: *On Simson-Wallace Theorem and Its Generalizations.* J. Geometry and Graphics **9** (2005). 141-153.

[71] P. Pech: *A Harmonic Analysis of Polygons.* Slov. časopis pro geometrii a grafiku **2** (2005). 17-36.

[72] P. Pech: *Od Napoleonovy věty k mozaikám.* Sborník 18. semináře pro geometrii a počítačovou grafiku, VUT Brno, JČMF, 1998, 55-61.

[73] D. Pedoe: *Geometry.* Dover Publ. New York 1988.

[74] K. Petr: *O jedné větě pro mnohoúhelníky rovinné.* Čas. pro pěst. mat. a fyz. **34** (1905), 166-172.

[75] M.A. Rashid, A.O. Ajibade: *Two conditions for a quadrilateral to be cyclic expressed in terms of the lengths of its sides.* Int. J. Math. Educ. Sci. Techn. **34**, No. 5, 2003, 739-742.

[76] T. Recio, H. Sterk, M.P. Vélez: *Project 1. Automatic Geometry Theorem Proving.* In: Some Tapas of Computer Algebra, A. Cohen, H. Cuipers, H. Sterk (eds), Algorithms and Computations in Mathematics, Vol. 4, Springer, 1998, 276-296.

[77] T. Recio, M.P. Vélez: *Automatic Discovery of Theorems in Elementary Geometry.* J. Automat. Reason. **12** (1998), 1-22.

[78] L. Rédey, B.Sz. Nagy: *Eine Verallgemeinerung der Inhaltsformel von Heron.* Publ. Math. Debrecen **1** (1949), 42-50.

[79] R. Riesinger: *On Wallace Loci from the Projective Point of View.* J. Geom. and Graphics **8** (2004), 201-213.

[80] J.F. Rigby: *Napoleon, Escher and Tessellations.* Structural Topology **17** (1991), 17-23.

[81] E. Roanes M., M. Roanes - Lozano: *Automatic Determination of Geometric Loci. 3D-Extension of Simson - Steiner Theorem.* J.A. Campbell and E. Roanes - Lozano (Eds.): AISC 2000, LNAI 1930, Springer 2001, 157-173.

[82] D.P. Robbins: *Areas of polygons inscribed in a circle.* Discrete Comput. Geom. **12** (1994), 223-236.

[83] E.J. Routh: *A Treatise on Analytical Statics, with Numerous Examples.* Vol. I, 2nd ed., Cambridge University Press, London 1896.

[84] S. Sadov: *Sadov's Cubic Analog of Ptolemy's Theorem.* http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/sadov.html

[85] I.J. Schoenberg: *The Harmonic Analysis of Skew Polygons as a Source of Outdoor Sculptures.* The Coxeter Festschrift, Geometric Vein, Springer, Berlin 1982, 165-176.

[86] P. Schreiber: *On the Existence and Constructibility of Inscribed Polygons.* Beiträge zur Algebra und Geometrie **34** (1993), 195-199.

[87] Ch.J. Scriba: *Wie kommt "Napoleon Satz" zu seinem Namen?* Hist. Math. **8** (1980), 458-459.

[88] G.C. Shephard: *Euler's Triangle Theorem.* Crux Math. **25** (1999), 148-153.

[89] G.C. Shephard: *Cyclic sums for polygons.* Math. Magazine **72** (1999), 126-132.

[90] G.C. Shephard: *The Compleat Ceva.* Math. Gazette **83** (1999), 1-8.

[91] M. Simon: *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert.* Teubner, Leipzig 1906.

[92] D.O. Škljarskij, N.N. Čencov, I.M. Jaglom: *Geometričeskije neravenstva i zadači na maximum i minimum.* Nauka, Moskva 1970.

[93] D.O. Škljarskij, N.N. Čencov, I.M. Jaglom: *Izbrannyje zadači i teoremy elementarnoj matematiky, časť 2 Geometrija (planimetrija).* Moskva 1952.

[94] S. Šmakal: *Eine Bemerkung zu einem Satz über räumliche Fünfecke.* Elem. der Math. **27** (1972), 62-63.

[95] Ch.R. Staudt: *Über die Inhalte der Polygone und Polyeder.* Journal für die reine und angewandte Mathematik **24** (1842), 252-256.

[96] D. Svrtan, D. Veljan, V. Volenec: *Geometry of pentagons: from Gauss to Robbins.* http://arxiv.org/abs/math/0403503

[97] A. Tarski: *The completness of elementary algebra and geometry.* Institut Blaise Pascal, Paris 1967, iv + 55pp. Reprinted from a paper which was prepared to printing in 1940 in publishing house Hermann & Cie but was not edited due to the World War the 2nd.

[98] V. Thébault: *Solution to Problem 169.* National Math. Mag. **12** (1937-38), 192-194.

[99] B.L. van der Waerden: *Ein Satz über räumliche Fünfecke.* Elem. der Math. **25** (1970), 73-96.

[100] B.L. van der Waerden: *Nachtrag zu "Ein Satz über räumliche Fünfecke"*. Elem. der Math. **27** (1972), 63.

[101] D. Wang: *Gröbner Bases Applied to Geometric Theorem Proving and Discovering.* In: Gröbner bases and applications. B. Buchberger, F. Winkler (eds), Lecture Notes of Computer Algebra, Cambridge Univ. Press, Cambridge, 1998, 281-301.

[102] D. Wang: *Elimination Practice. Software Tools and Applications.* Imperial College Press, London, 2004.

[103] J. Waser: Ph.D. thesis, Calif. Institute of Techn., Pasadena, 1944.

[104] H. Wieleitner: *Geschichte der Mathematik, II. Teil.*1911 - 1921, Russian translation 1958, 2. edition 1966.

[105] F. Winkler: *Gröbner Bases in Geometry Theorem Proving and Simplest Degeneracy Conditions.* Mathematica Pannonica **1** (1990), 15-32.

[106] W. Wu: *On the Decision Problem and the Mechanization of Theorem Proving in Elementary Geometry.* Scientia Sinica **21** (1978), 157-179.

[107] W. Wu: *Mechanical Theorem Proving in Geometries.* Texts and Monographs in Symbolic Computation, Springer, 1994.